Continuous optimization
PGE305

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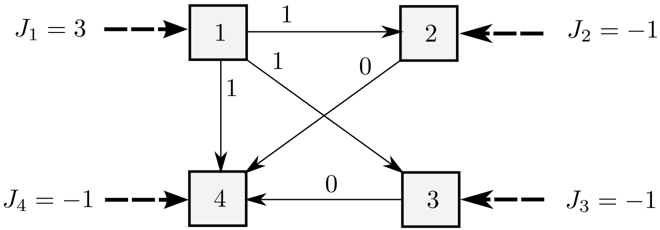
Exercise 8



Exercise 8

Edge e	Initial node $d(e)$	Terminal node $s(e)$	Resistance R_e
1	1	2	1
2	1	4	1
3	2	4	1
4	1	3	1
5	3	4	1

Definition: $\sigma(k) = \{e \mid s(e) = k\}$, $\mu(k) = \{e \mid d(e) = k\}$.

Constraints: $J_k + \sum_{e \in \sigma(k)} I_e = \sum_{e \in \mu(k)} I_e$, for all k.

$\rightarrow$ Kirchhoff's law. Explicite constraints:

$\sigma(1) = \emptyset$	$\mu(1) = \{1, 2, 4\}$
$\sigma(2) = \{1\}$	$\mu(2) = \{3\}$
$\sigma(3) = \{4\}$	$\mu(3) = \{5\}$
$\sigma(4) = \{2, 3, 5\}$	$\mu(4) = \emptyset$

$$\begin{aligned} J_1 &= I_1 + I_2 + I_4 \\ J_2 + I_1 &= I_3 \\ J_3 + I_4 &= I_5 \\ J_4 + I_2 + I_3 + I_5 &= 0 \end{aligned}$$

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Lagrangian:

$$\begin{aligned} L(I,V) = \tfrac{1}{2}(R_1 I_1^2 + \ldots + R_5 I_5^2) &+ V_1(I_1 + I_2 + I_4 - J_1) \\ &+ V_2(I_3 - I_1 - J_2) \\ &+ V_3(I_5 - I_4 - J_3) \\ &+ V_4(-I_2 - I_3 - I_5 - J_4). \end{aligned}$$

KKT conditions:

$$\begin{aligned} R_1 I_1 + V_1 - V_2 &= 0 \\ R_2 I_2 + V_1 - V_4 &= 0 \\ R_3 I_3 + V_2 - V_4 &= 0 \\ R_4 I_4 + V_1 - V_3 &= 0 \\ R_5 I_5 + V_3 - V_4 &= 0. \end{aligned}$$

Interpretation: Ohm's law.

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General case. Lagrangian:

$$L(I,V) = \frac{1}{2}\sum_{e=1}^{M} R_e I_e^2 - \sum_{k=1}^{N}\Big(\sum_{e\in\mu(k)} I_e - \sum_{e\in\sigma(k)} I_e - J_k\Big)V_k.$$

We have:

$$\sum_{k=1}^{N}\sum_{e\in\mu(k)} I_e V_k = \sum_{e=1}^{M} I_e V_{s(e)}$$

$$\sum_{k=1}^{N}\sum_{e\in\sigma(k)} I_e V_k = \sum_{e=1}^{M} I_e V_{d(e)}.$$

Lagrangian: $L(I,V) = \sum_{e=1}^{M}\Big(\frac{1}{2}R_e I_e^2 - I_e(V_{s(e)} - Vd(e))\Big) + \text{ Cte.}$

KKT conditions: $R_e I_e = V_{s(e)} - V_{d(e)}$.

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Similar principle for gas networks.

- I_e: massic flow rate in pipe e
- V_k: pression at node k
- R_e: coefficient characteristic from pipe e.

Kirchhoff's law + Darcy-Weisbach's law:

$$V_{s(e)} - V_{d(e)} = \begin{cases} R_e I_e^2 & \text{if } V_e \geq 0 \\ -R_e I_e^2 & \text{otherwise.} \end{cases}$$

Corresponding potential problem:

$$\inf \sum_{e=1}^{M} \frac{1}{3} R_e |I_e|^3 \quad \text{subject to: Kirchhoff's law.}$$