Optimal control of ODEs

Exercise sheet

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Exercise 1 We consider the following problem:

$$\inf_{\substack{T\geq 0\\ y\in W^{1,\infty}(0,T;\mathbb{R}^2)\\ u\in L^\infty(0,T;\mathbb{R})}} T, \quad \text{subject to}: \begin{cases} \dot{y}_1(t)=u(t)\\ \dot{y}_2(t)=-y_2(t)+u(t)\\ (y_1(0),y_2(0))=(x_1,x_2)\\ (y_1(T),y_2(T))=(0,0)\\ u(t)\in[-1,1], \end{cases}$$

for a given initial condition $(x_1,x_2)\in\mathbb{R}^2$. We will denote by $y[x,u]$ the solution to the costate equation for a given initial condition x and a given control u.

1. Let us assume the existence of a feasible triplet (T,y,u). Justify that the problem possesses a solution.
2. Let us fix a solution $(T,\bar{y},\bar{u})$ to the problem. Write Pontryagin's principle (we denote by $(\bar{p}_1,\bar{p}_2)$ the associated costate).
3. Find an explicit expression of $(\bar{p}_1,\bar{p}_2)$ in function of $(\bar{p}_1(T),\bar{p}_2(T))$.
4. Prove that $\bar{u}$ is piecewise constant, with at most two pieces, and that $\bar{u}(t)\in\{-1,1\}$ for a.e. $t\in[0,T]$.
5. Find an explicit expression of $y[x,u]$ for u constant equal to 1 and for u constant equal to -1.
6. Compute the following sets:

$$\Gamma_1=\big\{x\in\mathbb{R}^2\,|\,\exists T\geq 0,\ y[x,u=1](T)=(0,0)\big\},$$
$$\Gamma_{-1}=\big\{x\in\mathbb{R}^2\,|\,\exists T\geq 0,\ y[x,u=1](T)=(0,0)\big\}.$$

7. On a graph, draw (approximatively) Γ_1 and Γ_{-1}. Draw the optimal trajectories of the problem for a set of different initial conditions.

Exercise 2 We consider the following time-optimal control problem:

$$\inf_{\substack{T\geq 0\\ y\in W^{1,\infty}(0,T;\mathbb{R}^n)\\ u\in L^\infty(0,T;\mathbb{R}^m)}} T, \quad \text{s.t.:} \begin{cases} \dot{y}(t)= & Ay(t)+Bu(t),\\ y(0)\in & C_0,\ y(T)\in C_1,\ u(t)\in U. \end{cases} \tag{P}$$

The dynamics' coefficients $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times m}$ and the sets $U\subset\mathbb{R}^m$, $C_1\subset\mathbb{R}^n$, and $C_0\subset\mathbb{R}^n$ are given. We make the following assumptions: (i) U is convex, compact, and non-empty, (ii) C_0 is convex, compact, and non-empty, (iii) C_1 is convex, closed, and non-empty, (iv) the problem is feasible.

The fundamental difference with the situation of the lecture is that the initial condition is not fixed anymore, but can be freely chosen in C_0. Under the above assumptions, the existence of an optimal triplet $(T,\bar{y},\bar{u})$ can be established, with a natural extension of the proof seen in the lecture. It can also be shown that there exists $q\in N_{C_1}(\bar{y}(T))\backslash\{0\}$ such that the pair $(\bar{y},\bar{u})$ is a solution to the auxiliary problem the following auxiliary problem:

$$\inf_{\substack{y\in W^{1,\infty}(0,T;\mathbb{R}^n)\\ u\in L^\infty(0,T;\mathbb{R}^m)}} \langle q,y(T)\rangle, \quad \text{s.t.:} \begin{cases} \dot{y}(t)= & Ay(t)+Bu(t),\\ y(0)\in & C_0,\\ u(t)\in & U, \end{cases}$$

1. Formulate necessary and sufficient optimality conditions for the auxiliary problem. Prove them rigorously.

Exercise 3 Consider the following optimal control problem:

$$\inf_{\substack{u\in L^2(0,1)\\ y\in H^1(0,1)}} \int_0^1 \frac{1}{2}y(t)^2 + \frac{1}{6}u(t)^2\,\mathrm{d}t \quad \text{subject to:} \quad \begin{cases} \dot{y}(t) = & y(t)+u(t) \\ y(0) = & 1. \end{cases}$$

1. Calculate the pre-Hamiltonian and its derivatives.
2. Justify the existence of a unique solution to the problem.
3. Write the optimality conditions.
4. Let (y,u) denote the solution, with associated costate p. Let us set

   $$z_1(t) = y(t) - p(t) \quad \text{and} \quad z_2(t) = y(t) + 3p(t).$$

   Show that z_1 and z_2 are solutions to independent linear differential equations. Compute $z_1(t)$ and $z_2(t)$ in function of $z_1(0)$ and $z_2(0)$.
5. Compute y and p in function of $p(0)$.
6. Formulate the shooting equation and solve it.

Exercise 4 We consider the following problem:

$$\inf_{\substack{u\in L^2(0,T;\mathbb{R}^m)\\ y\in H^1(0,T;\mathbb{R}^n)}} \int_0^T \frac{1}{2}\langle y, Wy\rangle + \langle w, y\rangle + \frac{1}{2}\|u(t)\|^2\mathrm{d}t \quad \text{subject to:} \quad \begin{cases} \dot{y}(t) = & Ay(t)+Bu(t) \\ y(0) = & y_0. \end{cases}$$

Data: $T>0$, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $W\in\mathbb{R}^{n\times n}$, $w\in\mathbb{R}^n$, y_0. The matrix W is assumed to be symmetric positive semi-definite. Given $u\in L^2(0,T;\mathbb{R}^m)$, denote $y[u]$ the solution to the state equation. Let us denote by J the cost function of the optimal control problem, obtained after elimination of the state variable y.

We recall two inequalities, useful for this exercise:

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \quad \text{and} \quad (a+b)^2 \leq 2a^2 + 2b^2, \quad \forall a\in\mathbb{R},\ \forall b\in\mathbb{R}.$$

We recall that there exists a constant $M_1>0$ such that for all $u\in L^2(0,T;\mathbb{R}^m)$,

$$\|y[u]\|^2_{L^\infty(0,T;\mathbb{R}^n)} \leq M_1\big(\|y_0\|^2 + \|u\|^2_{L^2(0,T;\mathbb{R}^m)}\big).$$

1. Prove the existence of a constant $M_2>0$, independent of y_0 and w such that

   $$J(0) \leq M_2\big(\|y_0\|^2 + \|w\|^2\big).$$

2. Let $u\in L^2(0,T;\mathbb{R}^m)$ and let $y=y[u]$. Prove the existence of a constant $M_3>0$, depending on T only, such that

   $$J(u) \geq \frac{1}{2}\|u\|^2_{L^2(0,T;\mathbb{R}^m)} - \frac{M_3}{\varepsilon}\|w\|^2 - \varepsilon M_3\|y\|^2_{L^\infty(0,T;\mathbb{R}^n)}, \quad \forall\varepsilon>0.$$

3. Let $u\in L^2(0,T;\mathbb{R}^m)$ be such that $J(u)\leq J(0)$. Prove the existence of $M_4>0$, independent of u, y_0, and w, such that

   $$\|u\|^2_{L^2(0,T;\mathbb{R}^m)} \leq M_4\big(\|y_0\|^2 + \|w\|^2\big).$$

4. Prove the existence and uniqueness of a solution to the problem.

Exercise 5 This exercise is the continuation of the previous one. We denote by $\bar{u}$ the unique solution to the problem and by $\bar{y}$ the associated trajectory.

1. Calculate the Fréchet derivative of J. Find the Riesz representative of $DJ(\bar{u})$.
2. Write the optimality system associated with the problem. Denote by p the adjoint variable. The decoupling technique seen in the lecture can be extended to the new setting. To this purpose, we introduce a function $E\colon[0,T]\to\mathbb{R}^{n\times n}$ and we set:

   $$r = p + Ey.$$

   Find an expression of $\dot{r}$ which does not depend on p. Propose an appropriate Riccati equation for the variable E, so that the system can be decoupled.