1. INTRODUCTION

Our results so far were based on optimality conditions (Pontryagin’s principle).
Now: a different approach, based on dynamic programming.
In some sense, more specific to optimal control.
The dynamic programming principle is ubiquitous in optimization. A very general concept allowing to “split” some problems into a family of simpler problems.
The central tool: the value function $V$.
- Defined as the value of the optimization problem, expressed as a function of the initial state.
- Characterized as the unique viscosity solution of a non-linear partial differential equation (PDE) called HJB equation.

Problem formulation
Data of the problem and assumptions:
- A parameter $\lambda > 0$.
- A non-empty and compact subset $U$ of $\mathbb{R}^m$.
- A bounded and $L_f$-Lipschitz continuous mapping $f: (u, y) \in U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, i.e.
  \[ \| f \|_{\infty} := \sup_{(u, y) \in U \times \mathbb{R}^n} \| f(u, y) \| < \infty \]
  \[ \| f(u_2, y_2) - f(u_1, y_1) \| \leq L_f \| (u_2, y_2) - (u_1, y_1) \|, \]
  for all $(u_1, y_1)$ and $(u_2, y_2) \in U \times \mathbb{R}^n$.
- A bounded and $L_\gamma$-Lipschitz continuous mapping $\gamma: (u, y) \in U \times \mathbb{R}^n \rightarrow \mathbb{R}$.

Problem formulation
Notation: for any $\tau \in [0, \infty]$, $\mathcal{U}_\tau$ is the set of measurable functions from $(0, \tau)$ to $U$.
State equation: for $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_\infty$, there is a unique solution $y[u, x]$ to the ODE
\[ \dot{y}(t) = f(u(t), y(t)), \quad y(0) = x, \]
by the Picard-Lindelöf theorem (Cauchy-Lipschitz).
Cost function $W$, for $u \in \mathcal{U}_\infty$ and $x \in \mathbb{R}^n$:
\[ W(u, x) = \int_0^\infty e^{-\lambda t} \gamma(u(t), y[u, x](t)) \, dt. \]
Optimal control problem and value function $V$:
\[ V(x) = \inf_{u \in \mathcal{U}_\infty} W(u, x). \quad (P(x)) \]

Grönwall’s lemma
Lemma 1. (Grönwall’s lemma). Let $\alpha > 0$ and let $\beta > 0$. Let $\theta: [0, \infty) \rightarrow \mathbb{R}$ be a continuous function such that
\[ \theta(t) \leq \alpha + \beta \int_0^t \theta(s) \, ds, \quad \forall t \in [0, \infty). \]
Then, $\theta(t) \leq \alpha e^{\beta t}$, for all $t \in [0, \infty)$.
Corollary 2. Let $u \in \mathcal{U}_\infty$. For all $x$ and $\tilde{x}$, for all $t \geq 0$, it holds:
\[ \| y[u, x](t) - y[u, \tilde{x}](t) \| \leq e^{\frac{\beta}{2} t} \| x - \tilde{x} \|. \]
Proof. Grönwall’s lemma with $\theta = \| y[u, x] - y[u, \tilde{x}] \|$, $\alpha = \| x - \tilde{x} \|$, $\beta = L_f$.
2. DYNAMIC PROGRAMMING PRINCIPLE

[Dynamic programming principle]

Theorem 3. (Dynamic programming (DP) principle). Let \( \tau > 0 \). Then for all \( x \in \mathbb{R}^n \), abbreviating \( y = y[u,x] \),
\[
V(x) = \inf_{u \in \mathcal{U}} \left( \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, dt + e^{-\lambda \tau} V(y(\tau)) \right).
\]

(\textit{DPP})

Interpretation:
- \( V(x) \) is the value function of an optimal control problem on the interval \( (0, \tau) \).
- The original integral has been truncated:
\[
\int_0^\infty e^{-\lambda t} \ell(u(t), y(t)) \, dt \sim e^{-\lambda \tau} V(y(\tau)).
\]

The term \( e^{-\lambda \tau} V(y(\tau)) \) is the “optimal cost from \( \tau \) to \( \infty \).

[Flow property]

Lemma 4. (Flow property). Let \( x \in \mathbb{R}^n \) and let \( u \in \mathcal{U}_\infty \). Define:
- \( u_1 = u|_{(0, \tau)} \in \mathcal{U}_\tau \)
- \( u_2 = u|_{(\tau, \infty)} \in L^\infty(\tau, \infty; \mathcal{U}) \)
- \( \tilde{u}_2 \in \mathcal{U}_\infty, u_2(t) = \tilde{u}_2(t + \tau) \).

It holds:
\[
y[u,x](t) = y[\tilde{u}_2, y[u_1, x](\tau)](t - \tau),
\]
for any \( t \geq \tau \).

Remark. After time \( \tau \), one can forget \( u_1 \) and only remember \( y[x, u_1](\tau) \).

[Proof]

Proof of the DP-principle. Let us denote
\[
\tilde{V}(x) = \inf_{u \in \mathcal{U}} \left( \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, dt + e^{-\lambda \tau} V(y(\tau)) \right).
\]

Step 1: \( V \geq \tilde{V} \). Let \( u, u_1, u_2 \), and \( \tilde{u}_2 \) be as in Lemma 4.
\[
W(u, x) = \int_0^\infty e^{-\lambda t} \ell(u(t), y[u, x](t)) \, dt
\]
\[
= \int_0^\tau e^{-\lambda t} \ell(u(t), y[u, x](t)) \, dt \\
+ e^{-\lambda \tau} \int_\tau^\infty e^{-\lambda (t - \tau)} \ell(u(t), y[u, x](t)) \, dt \\
= \int_0^\tau e^{-\lambda t} \ell(u(t), y[u, x](t)) \, dt \\
+ e^{-\lambda \tau} \int_0^\infty e^{-\lambda s} \ell(u(s + \tau), y[u, x](s + \tau)) \, ds.
\]

[Proof]

We further have, for the last integral:
\[
\int_0^\infty e^{-\lambda s} \ell(u(s + \tau), y[u, x](s + \tau)) \, ds
\]
\[
= \int_0^\infty e^{-\lambda t} \ell(\tilde{u}_2(s), y[\tilde{u}_2, y[u_1, x](\tau)](s)) \, ds
\]
\[
= W(\tilde{u}_2, y[u_1, x](\tau)) \geq V(y[u_1, x](\tau)).
\]

Injecting in the above equality:
\[
W(u, x) \geq \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) \, dt \\
+ e^{-\lambda \tau} V(y[u_1, x](\tau)) \\
\geq \tilde{V}(x).
\]

Minimizing with respect to \( u \) yields \( V \geq \tilde{V} \).

[Proof]

Step 2: \( \tilde{V} \leq V \). Let \( \varepsilon > 0 \). Let \( u_1 \in \mathcal{U}_\tau \) be such that
\[
\int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) \, dt + e^{-\lambda \tau} V(y[u_1, x](\tau)) \\
\leq \tilde{V}(x) + \varepsilon / 2.
\]

Let \( \tilde{u}_2 \in \mathcal{U}_\infty \) be such that
\[
W(\tilde{u}_2, y[u_1, x](\tau)) \leq V(y[u_1, x](\tau)) + \varepsilon / 2.
\]

Let \( u \) be defined by
\[
u(t) = \begin{cases} 
  u_1(t) & \text{for a.e. } t \in (0, \tau), \\
  \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty).
\end{cases}
\]

[Proof]

The same calculation as above yields:
\[
W(u, x) = \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) \, dt \\
+ e^{-\lambda \tau} \int_0^\tau e^{-\lambda (t - \tau)} \ell(\tilde{u}_2(t), y[\tilde{u}_2(t), y[u_1, x](\tau)](t)) \, dt \\
= W(\tilde{u}_2, y[u_1, x](\tau)).
\]

Therefore,
\[
W(u, x) \leq \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) \, dt \\
+ e^{-\lambda \tau} (V(y[u_1, x](\tau)) + \varepsilon / 2)
\leq \tilde{V}(x) + \varepsilon.
\]

It follows that
\[
V(x) \leq \tilde{V}(x) + \varepsilon, \quad \forall \varepsilon > 0.
\]
3. A FIRST CHARACTERIZATION OF THE VALUE FUNCTION

**[Decoupling]**

**Corollary 5.** Let \( u \in \mathcal{U}_\infty \) be a solution to \( P(x) \).

Let \( \tau > 0 \). Let \( u_1 \) and \( \tilde{u}_2 \) be defined as in Lemma 4. Then,

- \( u_1 \) is optimal in the DP principle
- \( \tilde{u}_2 \) is optimal for \( P(y[u_1, x](\tau)) \).

**Proof.** Step 1: We have \( u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0, \tau) \\ \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). \end{cases} \)

Then \( u \) is a solution to \( P(x) \).

What can we do with the value function? If \( V \) is known, then the DP-principle allows to **decouple** the problem in time.

**[Regularity of V]**

**Lemma 6.** The value function \( V \) is bounded. It is also uniformly continuous, that is, for all \( \varepsilon > 0 \), there exists \( \alpha > 0 \) such that for all \( x \) and \( \tilde{x} \in \mathbb{R}^n \),

\[
\|\tilde{x} - x\| \leq \alpha \implies |V(\tilde{x}) - V(x)| \leq \varepsilon.
\]

**Proof.** Step 1: proof of boundedness. Let \( x \in \mathbb{R}^n \) and \( u \in \mathcal{U}_\infty \). We have

\[
|W(x, u)| \leq \int_0^\infty e^{-\lambda t} \|\ell\| \|\ell\| \ dt \leq \frac{1}{\lambda} \|\ell\| \|\ell\|.
\]

Thus \( |V(x)| \leq \frac{1}{\lambda} \|\ell\| \|\ell\| \).

**[More regularity of V]**

**Lemma 7.** We have

- if \( \lambda < L_f \), then \( V \) is \((\lambda/L_f)\)-Hölder continuous
- if \( \lambda = L_f \), then \( V \) is \( \alpha \)-Hölder continuous for all \( \alpha \in (0,1) \)
- if \( \lambda > L_f \), then \( V \) is Lipschitz continuous.

**DP-mapping**

**Notation:** \( \text{BUC}(\mathbb{R}^n) \) is the set of bounded and uniformly continuous functions from \( \mathbb{R}^n \) to \( \mathbb{R} \).

**Lemma 8.** The space \( \text{BUC}(\mathbb{R}^n) \), equipped with the uniform norm (denoted \( \| \cdot \|_\infty \)) is a Banach space.

Fix \( \tau > 0 \). Consider the “DP-mapping” (also called Bellman operator):

\[
T : v \in \text{BUC}(\mathbb{R}^n) \mapsto T v \in \text{BUC}(\mathbb{R}^n),
\]

defined by

\[
Tv(x) = \inf_{u \in \mathcal{U}_t} \left( \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \ dt + e^{-\lambda T} v(y(\tau)) \right),
\]

where \( y = y[u, x] \).
We fix now
We conclude that
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Proof.
Lemma 10.
The value function $\|x - x\| \leq 0 \implies \|v(\tilde{x}) - v(x)\| \leq \varepsilon/2$. Let $\alpha > 0$. Let $x$ and $\tilde{x} \in \mathbb{R}^n$ be such that $\|x - \tilde{x}\| \leq \alpha$. The value of $\alpha$ will be fixed later. For all $u \in U$, for all $t \in [0, \tau]$, we have
\[
\|y[u, \tilde{x}](t) - y[u, x](t)\| \leq e^{\varepsilon/\tau} \|x - \tilde{x}\| \leq e^{\varepsilon/\tau} \alpha.
\]
We have $|Tv(\tilde{x}) - Tv(x)| \leq \Delta_1 + \Delta_2$, with...

[DP-mapping]

\[
\begin{align*}
\Delta_1 &= \sup_{u \in \mathcal{U}} \left| \int_0^\tau e^{-\lambda \tau}(u(t), y(t)) dt \right. \\
&\quad - \left. \int_0^\tau e^{-\lambda \tau}(u(t), \tilde{y}(t)) dt \right|,
\Delta_2 &= \sup_{u \in \mathcal{U}} \left| e^{-\lambda \tau}v(\tilde{y}(\tau)) - e^{-\lambda \tau}v(y(\tau)) \right|
\end{align*}
\]
We fix now $\alpha = e^{-\lambda \tau} \min \left( \alpha_0, \frac{\varepsilon}{2\tau} \right)$. We have
\[
\Delta_1 \leq \tau \lambda e^{\tau \lambda \alpha} \leq \varepsilon/2 \quad \text{and} \quad \Delta_2 \leq \varepsilon/2,
\]
since $\|\tilde{y}(\tau) - y(\tau)\| \leq e^{\varepsilon/\tau} \alpha \leq \alpha_0$. Therefore,
\[
|Tv(\tilde{x}) - Tv(x)| \leq \varepsilon.
\]

[Min-plus linearity]

Notation. Given $v_1$ and $v_2 \in \text{BUC}(\mathbb{R}^n)$, we write $v_1 \preceq v_2$ if $v_1(x) \leq v_2(x)$ for all $x \in \mathbb{R}^n$. We define $\min(v_1, v_2) \in \text{BUC}(\mathbb{R}^n)$ by
\[
\min(v_1, v_2)(x) = \min(v_1(x), v_2(x)), \quad \forall x \in \mathbb{R}^n.
\]
Given $\alpha \in \mathbb{R}$, we define $v_1 + \alpha$ by $(v_1 + \alpha)(x) = v_1(x) + \alpha$.

Lemma 11. Let $v_1$ and $v_2 \in \text{BUC}(\mathbb{R}^n)$. Let $\alpha \in \mathbb{R}$. The map $T$ is monotone:
\[
v_1 \preceq v_2 \implies Tv_1 \preceq Tv_2
\]
and min-plus linear:
\[
\min(Tv_1, Tv_2) = T \min(v_1, v_2), \quad T(v + \alpha) = (Tv) + e^{-\lambda \tau} \alpha.
\]
Proof: exercise.

4. HJB EQUATION: THE CLASSICAL SENSE

[Hamiltonian]
We define the pre-Hamiltonian $H$ and the Hamiltonian $H$ by
\[
H(u, x, p) = \ell(u, x) + \langle p, f(u, x) \rangle, \quad \Phi(x, p) = \min_{u \in \mathcal{U}} H(u, x, p).
\]
Lemma 12. The mapping $H$ is continuous, concave with respect to $p$, and Lipschitz continuous with respect to $p$ with modulus $\|f\|_{\infty}$.
Proof. The pre-Hamiltonian $H$ is affine in $p$, thus concave in $p$. As an infimum of concave functions, $H$ is concave. We have:
\[
\|H(x, \tilde{p}) - H(x, p)\| \leq \sup_{u \in \mathcal{U}} \|H(u, x, \tilde{p}) - H(u, x, p)\|
\]
\[
\leq \sup_{u \in \mathcal{U}} |\langle \tilde{p} - p, f(u, x) \rangle| \leq \|\tilde{p} - p\| \cdot \|f\|_{\infty}.
\]

[A characterization of $V$]

Lemma 10. The value function $V$ is the unique solution of the fixed-point equation:
\[
Tv = v, \quad v \in \text{BUC}(\mathbb{R}^n).
\]
Proof.
- Existence: direct consequence of the DP principle ($V = TV$).
- Uniqueness: for any $v$ such that $v = Tv$, we have $\|v - V\|_{\infty} = \|Tv - TV\|_{\infty} \leq e^{-\lambda \tau} \|v - V\|_{\infty}$.
Thus $v = V$.

Remark: the dynamic programming principle entirely characterises the value function!

[Informal derivation]

Notation: $C^1(\mathbb{R}^n)$, the set of continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$.
Lemma 13. Let $\Phi \in C^1(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$, let $u \in \mathcal{U}_{\infty}$, let $y = y[u, x]$. Consider the mapping:
\[
\varphi: \tau \in [0, \infty) \mapsto \int_0^\tau e^{-\lambda \tau}(u(t), y(t)) dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x).
\]
Then $\varphi(0) = 0$ and $\varphi \in W^{1, \infty}(0, \infty)$ with
\[
\varphi(\tau) = e^{-\lambda \tau} \left( H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau)) \right).
\]
In particular: $\varphi(0) = H(u(0), x, \nabla \Phi(x)) - \lambda \Phi(x)$ (if $u$ is continuous at 0).
Informal derivation
Proof. To simplify, we only consider the case where \( u \) is continuous, so that \( y \) is \( C^1 \) and \( \varphi \) is \( C^1(\mathbb{R}^n) \). We have then:
\[
\varphi(\tau) = e^{-\lambda\tau} \ell(u(\tau), y(\tau)) + e^{-\lambda\tau} (\nabla \Phi(y(\tau)), \dot{y}(\tau)) - \lambda e^{-\lambda\tau} \Phi(y(\tau))
\]
\[
= e^{-\lambda\tau} \left[ \ell(u(\tau), y(\tau)) + (\nabla \Phi(y(\tau)), f(u(\tau), y(\tau))) \right] - \lambda e^{-\lambda\tau} \Phi(y(\tau))
\]
\[
= e^{-\lambda\tau} \left[ H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau)) \right].
\]

HJB in the classical sense

**Theorem 14.** Let \( x \in \mathbb{R}^n \). Assume that

- \( V \) is continuously differentiable in a neighborhood of \( x \)
- \( P(x) \) has a solution \( \bar{u} \) which is continuous at time 0.

Then, \( \lambda V(x) - H(x, \nabla V(x)) = 0 \), \( \bar{u}(0) \in \text{argmin}_{u \in U} H(u_0, x, \nabla V(x)) \).

**Proof.** Step 1. Let \( u_0 \in U \), let \( u \) be the constant control equal to \( u_0 \), let \( y = y[u, x] \). By the dynamic programming principle, we have:
\[
0 \leq \varphi(\tau) := \int_0^\tau e^{-\lambda \tau} \ell(u(t), y(t)) \, dt + e^{-\lambda \tau} V(y(\tau)) - V(x),
\]
for all \( \tau \). Since \( \varphi(0) = 0 \), we deduce from (\star) that:
\[
0 \leq \varphi(0) = H(u_0, x, \nabla V(x)) - \lambda V(x).
\]
Therefore,
\[
0 \leq H(u_0, x, \nabla V(x)) - \lambda V(x), \quad \forall u_0 \in U.
\]

**HJB in the classical sense**
**Remarks.**
Let \( Q(u, y) := H(u, y, \nabla V(y)) \) and assume that \( V \in C^2(\mathbb{R}^n) \).

- If the minimizer is unique in the following relation, we have a feedback law:
  \[
  \bar{u}(t) = \text{argmin}_U Q(\cdot, \dot{y}(t)).
  \]

- In some cases, one can show that \( \nabla V(\dot{y}(t)) = p(t) \), where \( p \) is defined by some adjoint equation → Pontryagin’s principle.

- In Reinforcement Learning, the approximation of \( Q \) is a central objective.

We will call the equation
\[
\lambda v(x) - H(x, \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n \quad \text{(HJB)}
\]
the Hamilton-Jacobi-Bellman equation, with unknown \( v: \mathbb{R}^n \to \mathbb{R} \).

**Remarks.**
- In general \( V \) is not differentiable → in which sense is the HJB equation to be understood?
- In Theorem 14, we have shown that \( \bar{u}(t) \in \text{argmin}_{u \in U} H(u_0, y[\bar{u}(t), x], \nabla V(y[\bar{u}(t), x])) \), (under restrictive assumptions). We will see next that this necessary condition is also sufficient.

**Theorem 16.** (Verification). Let us assume the assumptions of Theorem 14 hold for all \( x \in \mathbb{R}^n \), so that the HJB equation is satisfied in the classical sense. Let \( x \in \mathbb{R}^n \). Assume that there exists a control \( \bar{u} \) such that
\[
\bar{u}(t) \in \text{argmin}_{u \in U} H(u_0, y[\bar{u}(t), x], \nabla V(y[\bar{u}(t), x])),
\]
where \( \bar{y} = y[\bar{u}, x] \). Then \( \bar{u} \) is globally optimal.

**Proof.** Consider the function:
\[
\varphi(\tau) = \int_0^\tau e^{-\lambda \tau} \ell(\bar{u}(t), \dot{y}(t)) \, dt + e^{-\lambda \tau} V(\dot{y}(\tau)) - V(x).
\]
We have \( \varphi(0) = 0 \). Using (\star) and Theorem 14, we obtain:
\[
\varphi(\tau) = e^{-\lambda \tau} \left[ H(\bar{u}(\tau), y(\tau), \nabla V(y(\tau)) - V(\dot{y}(\tau)) \right] = e^{-\lambda \tau} \left[ H(\dot{y}(\tau), \nabla V(y(\tau)) - V(\dot{y}(\tau)) \right]
\]
\[
= 0.
\]
Thus \( \varphi \) is constant, equal to 0. Its limit is given by:
\[
0 = \int_0^\infty e^{-\lambda \tau} \ell(\bar{u}(t), \dot{y}(t)) \, dt - V(x) = W(x, \bar{u}) - V(x),
\]
proving the optimality of \( \bar{u} \).

**Corollary 15.** Let \( t \geq 0 \), assume that \( \bar{u} \) is continuous in a neighborhood of \( t \) and that \( V \) is \( C^1 \) in a neighborhood of \( y(t) \), where \( \bar{y} := y[\bar{u}, x] \). Then,
\[
\bar{u}(t) \in \text{argmin}_{u \in U} H(u_0, \bar{y}(t), \nabla V(\bar{y}(t))).
\]
5. HJB EQUATION: VISCOSITY SOLUTIONS

[Abstract PDE]
We consider an abstract PDE of the form:
\[ F(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n, \]
where \( F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \)
is continuous. It contains the HJB equation with
\[ F(x, v(x), \nabla v(x)) = \lambda v - \mathcal{H}(x, p). \]
Goal of the section: showing that \( V \) is a viscosity solution to the HJB equation.

[Viscosity solutions]
Definition 17. Let \( v : \mathbb{R}^n \to \mathbb{R} \). The following sets are called sub- and superdifferential, respectively:
\[ D^-v(x) = \left\{ p \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{v(y) - v(x) - \langle p, y-x \rangle}{|y-x|} \geq 0 \right\}, \]
\[ D^+v(x) = \left\{ p \in \mathbb{R}^n \mid \limsup_{y \to x} \frac{v(y) - v(x) - \langle p, y-x \rangle}{|y-x|} \leq 0 \right\}. \]
Exercise. Let \( v(x) = |x| \). Show that \( D^-v(0) = [-1, 1] \).

[Viscosity solutions]
Definition 20. Let \( v : \mathbb{R}^n \to \mathbb{R} \). We call \( v \) a viscosity supersolution if
\[ F(x, v(x), p) \geq 0, \quad \forall p \in D^-v(x) \]
or, equivalently, if for all \( \Phi \in C^1(\mathbb{R}^n) \) such that \( v - \Phi \) has a local minimum in \( x \),
\[ F(x, v(x), \nabla \Phi(x)) \geq 0. \]
We call \( v \) a viscosity solution if it is a sub- and a supersolution.

[Viscosity solutions]
Theorem 21. The value function \( V \) is a viscosity solution of the HJB equation.
Step 1: \( V \) is a subsolution. Let \( x \in \mathbb{R}^n \), let \( \Phi \in C^1(\mathbb{R}^n) \) be such that \( V - \Phi \) has a local maximizer in \( x \) and \( V(x) = \Phi(x) \).
We have to prove that
\[ \lambda v(x) - \mathcal{H}(x, \nabla \Phi(x)) \leq 0. \]
Let \( u_0 \in U \), let \( u \) be the constant control equal to \( u_0 \) and let \( y = y[u, x] \). By the DPP, we have:
\[ V(x) \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t)) \, dt + e^{-\lambda \tau} V(y(\tau)). \]
If \( \tau \) is sufficiently small, we have \( V(y(\tau)) \leq \Phi(y(\tau)) \).

[Viscosity solutions]
This implies that for \( \tau \) sufficiently small,
\[ 0 \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t)) \, dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x) =: \varphi(\tau). \]
Since \( \varphi(0) = 0 \), we deduce with \( (\star) \) that
\[ 0 \leq \varphi(0) = H(u_0, x, \nabla \Phi(x)) - \lambda V(x). \]
Minimizing with respect to \( u_0 \in U \), we obtain:
\[ 0 \leq H(x, \nabla \Phi(x)) - \lambda V(x), \]
as was to be proved.

[Viscosity solutions]
Step 2: \( V \) is a supersolution. Let \( x \in \mathbb{R}^n \), let \( \Phi \in C^1(\mathbb{R}^n) \) be such that \( V - \Phi \) has a local minimizer in \( x \) and such that \( V(x) = \Phi(x) \).
We have to prove that
\[ \lambda V(x) - \mathcal{H}(x, \nabla \Phi(x)) \leq 0. \]
It follows from the dynamic programming principle that for \( \tau > 0 \) small enough
\[ \Phi(x) \geq \inf_{u \in U} \int_0^\tau e^{-\lambda t} \ell(u(t), y[x, u](t)) \, dt + e^{-\lambda \tau} \Phi(y[x, u](\tau)), \]
\[ =: \varphi[u](\tau) \]

Abstract PDE
We consider an abstract PDE of the form:
\[ F(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n, \]
where \( F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \)
is continuous. It contains the HJB equation with
\[ F(x, v(x), \nabla v(x)) = \lambda v - \mathcal{H}(x, p). \]
Goal of the section: showing that \( V \) is a viscosity solution to the HJB equation.

Sub- and superdifferentials
Definition 17. Let \( v : \mathbb{R}^n \to \mathbb{R} \). The following sets are called sub- and superdifferential, respectively:
\[ D^-v(x) = \left\{ p \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{v(y) - v(x) - \langle p, y-x \rangle}{|y-x|} \geq 0 \right\}, \]
\[ D^+v(x) = \left\{ p \in \mathbb{R}^n \mid \limsup_{y \to x} \frac{v(y) - v(x) - \langle p, y-x \rangle}{|y-x|} \leq 0 \right\}. \]
Exercise. Let \( v(x) = |x| \). Show that \( D^-v(0) = [-1, 1] \).

Sub- and superdifferentials
Lemma 18. Let \( v : \mathbb{R}^n \to \mathbb{R} \) be continuous. Let \( p \in \mathbb{R}^n \).
\( p \in D^-v(x) \iff \) there exists \( \Phi \in C^1(\mathbb{R}^n) \) such that \( \nabla \Phi(x) = p \) and \( v - \Phi \) has a local minimum in \( x \).
\( p \in D^+v(x) \iff \) there exists \( \Phi \in C^1(\mathbb{R}^n) \) such that \( \nabla \Phi(x) = p \) and \( v - \Phi \) has a local maximum in \( x \).
Proof. The implication \( \iff \) is admitted. The implication \( \iff \) is left as an exercise.

Sub- and superdifferentials
Remark. In the above lemma, one can chose \( \Phi(x) = v(x) \) without loss of generality. Thus, we have:
\( (v - \Phi) \) has a local minimum in \( x \iff v - \Phi \) is nonnegative in a neighborhood of \( x \iff v \) is locally bounded from below by \( \Phi \).
\( (v - \Phi) \) has a local maximum in \( x \iff v - \Phi \) is nonpositive in a neighborhood of \( x \iff v \) is locally bounded from above by \( \Phi \).
Remark. If \( v \) is Fréchet differentiable at \( x \), then the sub- and superdifferential are equal to \( \{\nabla v(x)\} \).

Viscosity solutions
Definition 19. Let \( v : \mathbb{R}^n \to \mathbb{R} \). We call \( v \) a viscosity subsolution if
\[ F(x, v(x), p) \leq 0, \quad \forall x \in \mathbb{R}^n, \quad \forall p \in D^+v(x) \]
or, equivalently, if for all \( \Phi \in C^1(\mathbb{R}^n) \) such that \( v - \Phi \) has a local maximum in \( x \),
\[ F(x, v(x), \nabla \Phi(x)) \leq 0. \]
Thus by Lemma 13,

\[
0 \geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \ddot{\psi}[u](t) \, dt
\]

\[
= \inf_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} \left( H(u(t), y[u](t), \nabla \Phi(y[u](t))) - \lambda \Phi(y[u](t)) \right) \, dt
\]

\[
\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} \left( H(y[u](t), \nabla \Phi(y[u](t))) - \lambda \Phi(y[u](t)) \right) \, dt.
\]

We have \( \psi[u](0) = H(x, \nabla \Phi(x)) - \lambda V(x) \), in particular, \( \psi[u](0) \) does not depend on \( u \).

Let \( \varepsilon > 0 \). There exists (exercise!) \( \tau > 0 \) such that

\[
|\psi[u](t) - \psi[u](0)| \leq \varepsilon, \quad \forall t \in [0, \tau], \forall u \in \mathcal{U}_\infty.
\]

The previous inequality yields

\[
0 \geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau (\psi[u](0) - \varepsilon) \, dt
\]

\[
\geq \tau (H(x, \nabla \Phi(x)) - \lambda V(x) - \varepsilon).
\]

Dividing by \( \tau \) and sending \( \varepsilon \) to 0, we get the result.

**Theorem 22.** (Comparison principle). Let \( v_1 \) be a subsolution to the HJB equation. Let \( v_2 \) be a supersolution to the HJB equation. Then

\[
v_1(x) \leq v_2(x), \quad \forall x \in \mathbb{R}^n.
\]

**Proof:** admitted.

**Corollary 23.** The value function \( V \) is the unique viscosity solution.

**Proof.** By the comparison principle, any viscosity solution \( v \) is such that \( v \leq V \) and \( v \geq V \).