# SOD 311 — Time-optimal linear problems

**Laurent Pfeiffer (Inria and CentraleSupélec, University Paris-Saclay)**

**[Objectives]**
- *Goal:* controlling a dynamical system so as to **reach a target as fast as possible**.
- *Focus:* linear systems $\dot{y}(t) = Ay(t) + Bu(t)$.
- *Issues:* existence of a solution, optimality conditions, graph of feedback $\kappa$.

## 1. EXAMPLE: THE LUNAR LANDING PROBLEM

**[Model]** A spatial engine has the dynamics:

$$m\ddot{h}(t) = u(t), \quad \forall t \geq 0, \tag{1}$$

where:

| | |
|---|---|
| $m$ | mass of the engine |
| $h(t)$ | heigth of the engine at time $t$ |
| $u(t)$ | propulsion force at time $t$ |
| $v(t) = \dot{h}(t)$ | velocity at time $t$. |

*Problem:* given $h_0$ and $v_0$, find the smallest $T > 0$ for which there exist time functions $h$ and $u$ satisfying (1), $(h(0), v(0)) = (h_0, v_0)$, and $(h(T), v(T)) = (0, 0)$.

**[Mathematical problem]** For simplicity, we take $m = 1$. We consider constraints on $u$.
Given $(h_0, v_0)$, the problem writes:

$$\inf_{\substack{T \geq 0 \\ h\colon [0,T]\to\mathbb{R} \\ v\colon [0,T]\to\mathbb{R} \\ u\colon [0,T]\to\mathbb{R}}} T, \quad \begin{cases} \dot{h}(t) = v(t), & h(0) = h_0, \quad h(T) = 0, \\ \dot{v}(t) = u(t), & v(0) = v_0, \quad v(T) = 0, \\ u(t) \in [-1, 1]. \end{cases}$$

*Remark.* The state $(h, v)$ is uniquely defined by the control $u$ (via the dynamical system).
For the moment: no theoretical tool at hand... let's see what we can do!

**[Accelerating trajectories]** For $u = 1$, we have

$$\begin{cases} v(t) = v_0 + t \\ h(t) = h_0 + tv_0 + \dfrac{1}{2}t^2. \end{cases}$$

We can isolate $t$ in the first line: $t = v(t) - v_0$ and inject the result in the second line:

$$h(t) = h_0 + (v(t) - v_0)v_0 + \frac{1}{2}(v(t) - v_0)^2.$$

The curve

$$\{(h(t), v(t)) \,|\, t \geq 0\}$$

is the portion of a **parabola**.

**[Accelerating trajectories]**

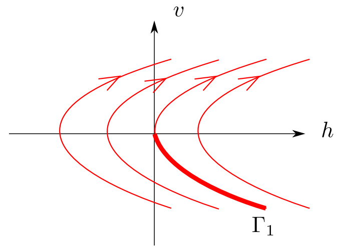



Fig. 1. Trajectories for $u = 1$ (acceleration).

**[Accelerating trajectories]** Let $\Gamma_1$ denote the set of initial conditions for which $u = 1$ steers $(h, v)$ to $(0, 0)$. We have: $(h_0, v_0) \in \Gamma_1 \iff$

$$\begin{cases} \exists T \geq 0 \\ 0 = v_0 + T \\ 0 = h_0 + Tv_0 + \dfrac{1}{2}T^2 \end{cases} \iff \begin{cases} v_0 \leq 0 \\ 0 = h_0 - v_0^2 + \dfrac{1}{2}v_0^2. \end{cases}$$

Therefore, $\Gamma_1 = \left\{ (h_0, v_0) \in \mathbb{R}^2 | \, v_0 \leq 0, \, h_0 = \frac{1}{2}v_0^2. \right\}$.

**[Decelerating trajectories]** For $u = -1$, we have

$$\begin{cases} v(t) = v_0 - t \\ h(t) = h_0 + tv_0 - \dfrac{1}{2}t^2. \end{cases}$$

We can isolate $t$ in the first line: $t = v_0 - v(t)$ and inject the result in the second line:

$$h(t) = h_0 + (v_0 - v(t))v_0 - \frac{1}{2}(v_0 - v(t))^2.$$

The curve $\{(h(t), v(t)) \,|\, t \geq 0\}$ is the portion of a **parabola**.

**[Decelerating trajectories]**

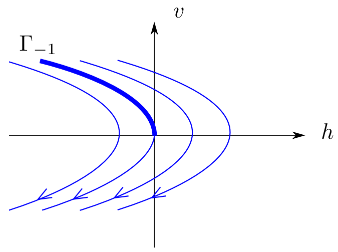



Fig. 2. Trajectories for $u = -1$ (deceleration).

**[Decelerating trajectories]** Let $\Gamma_{-1}$ denote the set of initial conditions for which $u = -1$ steers $(h, v)$ to $(0, 0)$. We have: $(h_0, v_0) \in \Gamma_{-1} \iff$

$$\begin{cases} \exists T \geq 0 \\ 0 = v_0 - T \\ 0 = h_0 + T v_0 - \frac{1}{2}T^2 \end{cases} \iff \begin{cases} v_0 \geq 0 \\ 0 = h_0 + v_0^2 - \frac{1}{2}v_0^2. \end{cases}$$

Therefore,

$$\Gamma_{-1} = \left\{ (h_0, v_0) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} v_0 \geq 0 \\ h_0 = -\frac{1}{2}v_0^2 \end{array} \right\}.$$

**[A simple case]** Consider the case $v_0 = 0$.
Then we should (fully) accelerate and (fully) decelerate on equal intervals of time.

- If $h_0 < 0$: accelerate $(u = 1)$ until $h(t) = h_0/2$, then decelerate $(u = -1)$.
- If $h_0 > 0$: decelerate $(u = -1)$ until $h(t) = h_0/2$, then accelerate $(u = 1)$.

**[A simple case]**

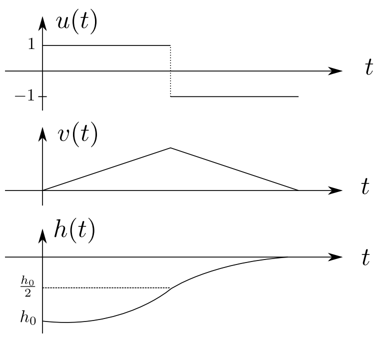



Fig. 3. Optimal control and trajectory for $v_0 = 0$ and $h_0 < 0$.

**[A simple case]**

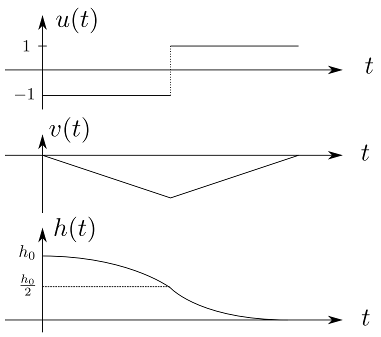



Fig. 4. Optimal control and trajectory for $v_0 = 0$ and $h_0 > 0$.

**[A simple case]**

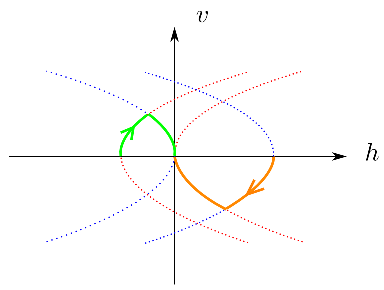



Fig. 5. Some optimal trajectories with null initial speed.

**[General case]** The theory (developed in the next sections) tells us the following.
For any $(h_0, v_0) \in \mathbb{R}^2$,

- There exists an optimal time $\bar{T}$ and an optimal control $\bar{u}$.
- Any optimal control takes values in $\{-1, 1\}$.
- Any optimal control is **piecewise constant**, with **atmost two pieces**.

**[General case]** In other words, for any optimal control $\bar{u}$, one of the following cases is satisfied:

(1) $\bar{u}(t) = 1$, for almost every $t \in (0, \bar{T})$
(2) $\bar{u}(t) = -1$, for a.e. $t \in (0, \bar{T})$
(3) "Accelerate-Decelerate": $\exists \tau \in (0, \bar{T})$ such that: $\bar{u}(t) = 1$, for a.e. $t \in (0, \tau)$, $\bar{u}(t) = -1$, for a.e. $t \in (\tau, \bar{T})$.
(4) "Decelerate-Accelerate": $\exists \tau \in (0, \bar{T})$ such that: $\bar{u}(t) = -1$, for a.e. $t \in (0, \tau)$, $\bar{u}(t) = 1$, for a.e. $t \in (\tau, \bar{T})$.

In the last two cases, $\tau$ is called **switching time**.
Remark for French readers: we use the english notation $(a, b)$ for the open interval, instead of the french notation $]a, b[$.

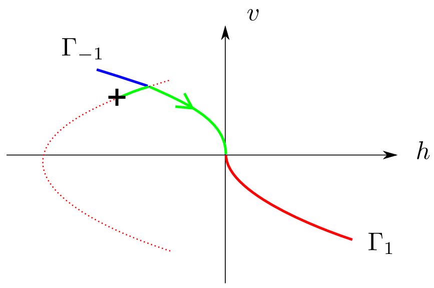
**[General case]** The problem is reduced to a **geometric** problem.
Find all trajectories such that...

- starting at the initial condition,
- ending up at the origin,
- made of two portions of parabola (a "red" and a "blue" one).

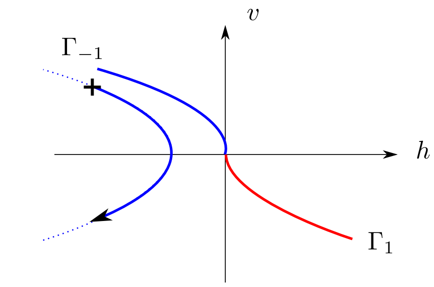
We will call them **Pontryagin** trajectories.
*Methodology:* for each initial condition,

- find all possible Pontryagin trajectories,
- find out the optimal one (there may exist Pontryagin trajectories which are not optimal).

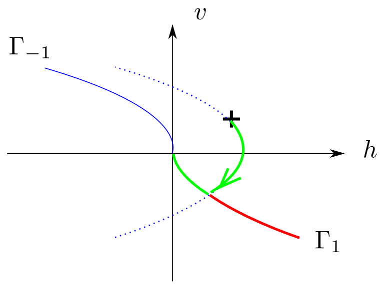
**[General case] First case:** $(h_0, v_0)$ lies strictly under $\Gamma_1 \cup \Gamma_{-1}$.
One possibility for the scenario "accelerate-decelerate".



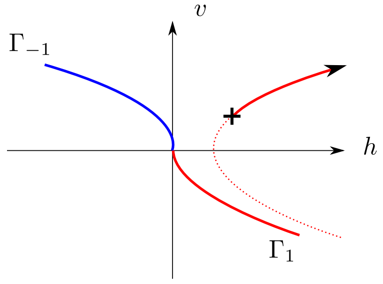

**[General case] First case:** $(h_0, v_0)$ lies strictly under $\Gamma_1 \cup \Gamma_{-1}$.
Zero possibility for the scenario "decelerate-accelerate".




**[General case] Second case:** $(h_0, v_0)$ lies strictly above $\Gamma_1 \cup \Gamma_{-1}$.
One possibility for the scenario "decelerate-accelerate".




**[General case] Second case:** $(h_0, v_0)$ lies strictly above $\Gamma_1 \cup \Gamma_{-1}$. Zero possibility for the scenario "accelerate-decelerate".




**[General case]** *Conclusion:* Whatever the initial condition, there is exactly one Pontryagin trajectory, which is necessarily optimal.

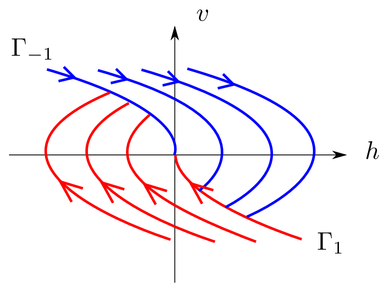



Fig. 6. Phase portrait of optimal trajectories.

**[General case]** We finally obtain a relation in **feedback form** for optimal controls $\bar{u}$ with associated trajectory $(\bar{h}, \bar{v})$:

$$\bar{u}(t) = \kappa(\bar{h}(t), \bar{v}(t)),$$

where $\kappa$ is defined by:

$$\kappa(h, v) = \begin{cases} 1 & \text{if } (h, v) \in \Gamma_1 \\ -1 & \text{if } (h, v) \in \Gamma_{-1} \\ 1 & \text{if } (h, v) \text{ lies strictly under } \Gamma_{-1} \cup \Gamma_1 \\ -1 & \text{if } (h, v) \text{ lies strictly above } \Gamma_{-1} \cup \Gamma_1, \end{cases}$$

for any $(h, v) \in \mathbb{R}^2 \backslash \{0\}$.
*Remark:* The feedback relation holds whatever the initial condition of the problem.

**[Summary]** The three main steps of our methodology:

- Calculation of **trajectories with constant controls** (with extremal values).
- Theory $\rightarrow$ **structural properties** of optimal controls.
- Reformulation of the problem as a **geometric** problem.

## 2. EXISTENCE OF A SOLUTION

**[Framework]** A general linear time-optimal control problem:

$$\inf_{\substack{T \geq 0 \\ y \in W^{1,\infty}(0,T;\mathbb{R}^n) \\ u \in L^\infty(0,T;\mathbb{R}^m)}} T, \quad \text{s.t.:} \begin{cases} \dot{y}(t) = Ay(t) + Bu(t), \\ y(0) = y_0, \\ y(T) \in C, \\ u(t) \in U. \end{cases} \tag{P}$$

Data of the problem and assumptions:

- Initial condition: $y_0 \in \mathbb{R}^n$
- Dynamics' coefficients: $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$
- A control set: $U \subset \mathbb{R}^m$, assumed convex, compact, non-empty
- A target: $C \subset \mathbb{R}^n$, assumed convex, closed, non-empty.

**[Matrix exponential]**

*Definition 1.* Let $M \in \mathbb{R}^{n\times n}$. We call **matrix exponential** $e^M$ the matrix
$$e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k \in \mathbb{R}^{n\times n}.$$

*Lemma 2.*
- For any operator norm $\|\cdot\|$, we have $\|e^M\| \leq e^{\|M\|}$.
- For all $t \in \mathbb{R}$, we have $\frac{d}{dt}e^{tM} = Me^{tM} = e^{tM}M$.
- Given $x_0 \in \mathbb{R}^n$, let $x\colon [0,\infty) \to \mathbb{R}^n$ be the solution to
$$\dot{x}(t) = Mx(t), \quad x(0) = x_0.$$
Then $x(t) = e^{tM}x_0$, for all $t \geq 0$.

**[State equation]** A pair $(y,u) \in W^{1,\infty}(0,T;\mathbb{R}^n) \times L^\infty(0,T;\mathbb{R}^m)$ satisfies the **state equation**: $\dot{y}(t) = Ay(t) + Bu(t)$, $y(0) = y_0$ if and only if
$$y(t) = y_0 + \int_0^t \Big(Ay(s) + Bu(s)\Big)\mathrm{d}s, \quad \forall t \in [0,T]. \tag{2}$$

*Theorem 3.* (Picard-Lindelöf / FR: Cauchy-Lipschitz). Given $y_0 \in \mathbb{R}^n$ and $u \in L^\infty(0,T;\mathbb{R}^m)$, there exists a unique $y$ satisfying (2). Moreover,
$$y(t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}Bu(s)ds. \quad \text{[Duhamel]}$$
*Notation:* $y[u]$.

**[Reachable set]** Some notation:
- $L^\infty(0,T;U)$: set of measurable functions from $(0,T)$ to $U$,
- $\bar{T}$: the value of problem $(P)$ ($\bar{T} = \infty$ if $(P)$ is infeasible).

*Definition 4.* Given $t \geq 0$, the **reachable set** at time $t$, $\mathcal{R}(t)$, is defined by
$$\mathcal{R}(t) = \big\{ y[u](t) \,|\, u \in L^\infty(0,t;U) \big\}.$$

*Lemma 5.*
- For all $T \geq 0$, the set $\cup_{0\leq t\leq T}\mathcal{R}(t)$ is **bounded**.
- For all $t \geq 0$, the reachable set $\mathcal{R}(t)$ is **convex**.

*Proof.* Exercise (use Duhamel's formula and boundedness of $U$).

**[Weak compactness]**

*Definition 6.* Let $F$ be a Banach space. Let $(e_k)_{k\in\mathbb{N}}$ be a sequence in $F$. The sequence **converges weakly** to $\bar{e} \in F$ (notation: $e_k \rightharpoonup \bar{e}$) if
$$L(e_k) \to L(\bar{e}),$$
for all continuous and linear map $L\colon F \to \mathbb{R}$.

*Remark.* If $e_k \rightharpoonup \bar{e}$, then $L(e_k) \to L(\bar{e})$ for any continuous and linear map $L\colon F \to \mathbb{R}^k$.

*Lemma 7.* Let $E$ be a closed and convex subset of a Hilbert space $F$. Let $(e_k)_{k\in\mathbb{N}}$ be a bounded sequence in $E$. Then there exists a weakly convergent subsequence $(e_{k_q})_{q\in\mathbb{N}}$ with weak limit in $E$.

*Proof.* See Corollary 3.22 and Proposition 5.1 in *Functional Analysis*, by H. Brézis.

**[Closedness of the reachable set]**

*Lemma 8.* (Closedness lemma). Let $(\tau_k)_{k\in\mathbb{N}}$ be a convergent sequence of positive real numbers with limit $\bar{\tau} \geq 0$. Assume that $\tau_k \geq \bar{\tau}$, $\forall k \in \mathbb{N}$.
Let $(y_k)_{k\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}^n$ with limit $\bar{y}$. Assume that
$$y_k \in \mathcal{R}(\tau_k), \quad \forall k \in \mathbb{N}.$$
Then $\bar{y} \in \mathcal{R}(\bar{\tau})$.

*Corollary 9.* For all $t \geq 0$, the set $\mathcal{R}(t)$ is **closed**.

**[Proof of the closedness lemma]**

*Proof. Step 1.* For all $k \in \mathbb{N}$, let $u_k \in L^\infty(0,\tau_k;U)$ be such that $y[u_k](\tau_k) = y_k$. As a consequence of Lemma 5, there exists $M > 0$ (independent of $k$) such that
$$\|\dot{y}[u_k]\|_{L^\infty(0,\tau_k;\mathbb{R}^m)} \leq M.$$
Thus $y[u_k](\cdot)$ is $M$-Lipschitz, that is
$$\|y[u_k](t_2) - y[u_k](t_1)\| \leq M|t_2 - t_1|, \quad \forall t_1,\, t_2 \in [0,T].$$
Next, we have $\|y[u_k](\bar{\tau}) - \bar{y}\| \leq$
$$\underbrace{\|y[u_k](\bar{\tau}) - y[u_k](\tau_k)\|}_{\leq M|\tau_k - \bar{\tau}|} + \|\underbrace{y[u_k](\tau_k)}_{y_k} - \bar{y}\| \to 0.$$
Thus $y[u_k](\bar{\tau}) \to \bar{y}$.

**[Proof of the closedness lemma]**

*Step 2.* Consider the linear map $L\colon u \in L^2(0,\bar{\tau};\mathbb{R}^m) \to \mathbb{R}^n$ defined by
$$L(u) = \int_0^{\bar{\tau}} e^{(\bar{\tau}-s)A}Bu(s)\,\mathrm{d}s.$$
By Cauchy-Schwarz inequality, we have
$$|L(u)| \leq \int_0^{\bar{\tau}} e^{(\bar{\tau}-s)\|A\|} \cdot \|B\| \cdot \|u(s)\|\,\mathrm{d}s$$
$$\leq \underbrace{\|B\| \cdot \Big(\int_0^{\bar{\tau}} e^{2(\bar{\tau}-s))\|A\|}\,\mathrm{d}s\Big)^{1/2}}_{<\infty} \|u\|_{L^2(0,\bar{\tau};\mathbb{R}^m)}.$$
This proves that the linear form $L$ is **continuous**.

**[Proof of the closedness lemma]**

*Step 3.* Apply Lemma 7:
- $L^2(0,\bar{\tau};\mathbb{R}^m)$ is a Hilbert space
- $L^\infty(0,\bar{\tau};U)$ is convex, closed, and bounded.

Then the sequence $u_k$ (restricted to $(0,\bar{\tau})$) has a weakly convergent subsequence, with limit $\bar{u}$.
We have:
$$\underbrace{y[u_{k_q}](\bar{\tau})}_{\longrightarrow \bar{y}} = e^{\bar{\tau}A}y_0 + \int_0^{\bar{\tau}} e^{(\bar{\tau}-s)A}Bu_{k_q}(s)ds$$
$$= e^{\bar{\tau}A}y_0 + L(u_{k_q}) \longrightarrow e^{\bar{\tau}A}y_0 + L(\bar{u}) = y[\bar{u}](\bar{\tau}),$$
proving that $\bar{y} = y[\bar{u}](\bar{\tau}) \in \mathcal{R}(\bar{\tau})$.

**[Existence result]**

*Theorem 10.* Assume that $\bar{T} < \infty$. There exists an optimal control, that is, there exists $\bar{u}$ such that

$$y[\bar{u}](\bar{T}) \in C.$$

*Proof.* Consider the set of times at which the target can be reached, that is:

$$\mathcal{T} = \big\{ T \geq 0 \,|\, \mathcal{R}(T) \cap C \neq \emptyset \big\}.$$

By assumption $\mathcal{T}$ is non empty. By definition, $\bar{T} = \mathit{inf}\, \mathcal{T}$.
Our task: proving that $\bar{T} \in \mathcal{T}$.

**[Existence result]**

- It suffices to show that $\mathcal{R}(\bar{T}) \cap C \neq \emptyset$.
- Let $\tau_k \downarrow \bar{T}$ be such that for all $k \in \mathbb{N}$, there exists $y_k \in \mathcal{R}(\tau_k) \cap C$. By Lemma 5, $(y_k)_{k \in \mathbb{N}}$ is bounded. Thus it has an **accumulation point** $\bar{y}$.
- Since $C$ is closed, $\bar{y} \in C$. By Lemma 8, $\bar{y} \in \mathcal{R}(\bar{T})$.

## 3. OPTIMALITY CONDITIONS

**[Methodology]** For proving the optimality conditions (in the form of a Pontryagin's principle), we proceed as follows:

- Fix an optimal control $\bar{u}$ for the time-optimal problem.
- Show that $\bar{u}$ is optimal for another problem, easier to treat, referred to as auxiliary problem.
- Establish Pontryagin's principle for the auxiliary problem.

### *3.1 Separation*

**[Hahn-Banach lemma]**

*Lemma 11.* Let $C_1$ and $C_2$ be two closed and convex sets of $\mathbb{R}^n$, let $C_2$ be bounded. Assume that $C_1 \cap C_2 = \emptyset$. Then, there exists $q \in \mathbb{R}^n \backslash \{0\}$ such that

$$\langle q, y_1 \rangle \leq \langle q, y_2 \rangle, \quad \forall y_1 \in C_1, \ \forall y_2 \in C_2.$$

We say that $q$ **separates** $C_1$ and $C_2$.

*Proof.* See Brezis, Theorem 1.7.

*Remark.* With loss of generality, we can assume that $\|q\| = 1$.

**[Hahn-Banach lemma]**

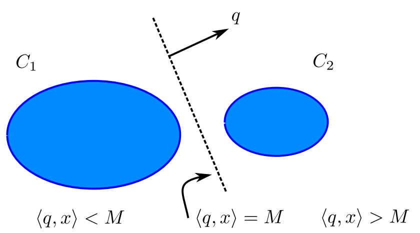



Fig. 7. Illustration of Hahn-Banach lemma.

**[Normal cones]**

*Definition 12.* Let $K$ be a subset of $\mathbb{R}^n$ and let $x \in K$. The normal cone of $K$ at $x$, denoted $N_K(x)$ is defined by

$$N_K(x) = \big\{ q \in \mathbb{R}^n \,|\, \langle q, y - x \rangle \leq 0, \ \forall y \in K \big\}.$$

*Some examples.*

- If $K = \{\bar{x}\}$, then $N_K(\bar{x}) = \mathbb{R}^n$.
- If $K = \mathbb{R}^n$, then $N_K(x) = \{0\}$ for any $x \in \mathbb{R}^n$.
- Let $\mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n \,|\, x_i \geq 0, i = 1, ..., n\}$.
  Let $\mathbb{R}^n_{\leq 0} := \{x \in \mathbb{R}^n \,|\, x_i \leq 0, i = 1, ..., n\}$. Then
  $$N_{\mathbb{R}^n_{\geq 0}}(0) = \mathbb{R}^n_{\leq 0} \quad \text{and} \quad N_{\mathbb{R}^n_{\leq 0}}(0) = \mathbb{R}^n_{\geq 0}$$

**[Normal cones]**

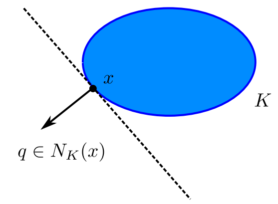



Fig. 8. A vector in the normal cone.

**[A separation result]**

*Lemma 13.* (Separation lemma). Let $\bar{T}$ denote the value of the time optimal control problem $(P)$. Assume that $0 < \bar{T} < \infty$. Then, there exists $\bar{q} \in \mathbb{R}^n \backslash \{0\}$ such that

$$\langle \bar{q}, z \rangle \leq \langle \bar{q}, y \rangle, \quad \forall z \in C, \quad \forall y \in \mathcal{R}(\bar{T}).$$

*Corollary 14.* For any **optimal control** $\bar{u}$, we have $\bar{q} \in N_C(y[\bar{u}](\bar{T}))$.

*Proof of the corollary.* Take $y = y[\bar{u}](\bar{T})$ in the separation lemma.

**[A separation result]**

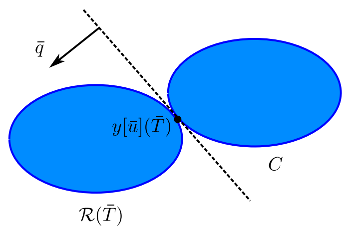



Fig. 9. Illustration of the separation lemma.

**[A separation result]** *Proof of the separation lemma.*

- Let $T_k \uparrow \bar{T}$. For all $k \in \mathbb{N}$, $\mathcal{R}(T_k) \cap C = \emptyset$.
- The set $C$ is convex and closed, $\mathcal{R}(T_k)$ is compact and convex (by Lemma 5 and Lemma 8).
- By the Hahn-Banach Lemma, there exists $q_k$ such that $\|q_k\| = 1$ and
$$\langle q_k, z \rangle \leq \langle q_k, y \rangle, \quad \forall z \in C, \ \forall y \in \mathcal{R}(T_k). \qquad (3)$$
Extracting a subsequence if necessary, we assume that $q_k \to \bar{q}$ for some $\bar{q} \in \mathbb{R}^n$ with $\|\bar{q}\| = 1$.

**[A separation result]** We next show that $\bar{q}$ separates $C$ and $\mathcal{R}(\bar{T})$.

- Let $z \in C$ and let $y \in \mathcal{R}(\bar{T})$.
Let $u \in L^\infty(0,T;U)$ be such that $y[u](\bar{T}) = y$.
Set $y_k = y[u](T_k) \in \mathcal{R}(T_k)$.
- Inequality (3) yields:
$$\langle q_k, z \rangle \leq \langle q_k, y_k \rangle, \quad \forall k \in \mathbb{N}.$$
- We pass to the limit and obtain
$$\langle \bar{q}, z \rangle \leq \langle \bar{q}, y \rangle.$$

*3.2 An auxiliary problem*

**[An auxiliary problem]** Let $T > 0$, let $y_0 \in \mathbb{R}^n$, and let $q \in \mathbb{R}^n$ be fixed.
Consider the following **auxiliary** optimal control problem:

$$\inf_{\substack{y \in W^{1,\infty}(0,T;\mathbb{R}^n) \\ u \in L^\infty(0,T;U)}} \langle q, y(T) \rangle, \quad \text{s.t.:} \begin{cases} \dot{y}(t) = Ay(t) + Bu(t) \\ y(0) = y_0, \\ u(t) \in U. \end{cases} \qquad (P_{\text{aux}}[q,T])$$

*Remark:* Let $(\bar{u}, \bar{y}, \bar{T})$ be a solution to the time-optimal problem. Let $\bar{q}$ be as in the separation lemma. Then $(\bar{u}, \bar{y})$ is a solution to $P_{\text{aux}}[q,T]$, with $(q,T) = (\bar{q}, \bar{T})$.

**[Pre-Hamiltonian and adjoint equation]** Define the **pre-Hamiltonian**:
$$H \colon (u,y,p) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \langle p, Ay + Bu \rangle \in \mathbb{R}.$$
Note that
$$H(u,y,p) = \langle A^\top p, y \rangle + \langle B^\top p, u \rangle.$$
Thus,
$$\nabla_y H(u,y,p) = A^\top p \quad \text{and} \quad \nabla_u H(u,y,p) = B^\top p.$$
Let us define $p$ as the solution to the **adjoint equation** (also called costate equation):
$$\begin{cases} p(T) = q \\ -\dot{p}(t) = A^\top p(t) = \nabla_y H(p(t)). \end{cases} \qquad (4)$$

**[Pontryagin's principle]**
*Theorem 15.* (Pontryagin's minimum principle). Let $(\bar{y}, \bar{u})$ be such that $\bar{y} = y[\bar{u}]$.
Then $(\bar{y}, \bar{u})$ is a **solution** to $(P_{\text{aux}}[q,T])$ if and only if
$$\bar{u}(t) \in \operatorname*{argmin}_{v \in U} H(v, \bar{y}(t), p(t)), \quad \text{for a.e. } t \in (0,T).$$
*Remark:*
$$\operatorname*{argmin}_{v \in U} H(v, \bar{y}(t), p(t)) = \operatorname*{argmin}_{v \in U} \langle B^\top p(t), v \rangle.$$

**[Proof of Pontryagin's principle]** "$\Longleftarrow$" Assume that $(\bar{y}, \bar{u})$ satisfies Pontryagin's principle.
Let $(y,u)$ be such that $y = y[u]$. Then
$$\begin{aligned}
&\langle q, y(T) - \bar{y}(T) \rangle \\
&\quad = \langle p(T), y(T) - \bar{y}(T) \rangle - \langle p(0), \underbrace{y(0) - \bar{y}(0)}_{= y_0 - y_0 = 0} \rangle \\
&\quad = \int_0^T \frac{d}{dt} \langle p(t), y(t) - \bar{y}(t) \rangle \, \mathrm{d}t \\
&\quad = \int_0^T \langle \dot{p}(t), y(t) - \bar{y}(t) \rangle + \int_0^T \langle p(t), \dot{y}(t) - \dot{\bar{y}}(t) \rangle \, \mathrm{d}t \\
&\quad = \int_0^T \langle -p(t), Ay(t) - A\bar{y}(t) \rangle \, \mathrm{d}t \\
&\qquad + \int_0^T \langle p(t), Ay(t) + Bu(t) - A\bar{y}(t) - B\bar{u}(t) \rangle \, \mathrm{d}t \\
&\quad = \int_0^T \langle B^\top p(t), u(t) - \bar{u}(t) \rangle \, \mathrm{d}t \geq 0.
\end{aligned}$$

**[Proof of Pontryagin's principle]** "$\Longrightarrow$" Assume that $(\bar{y}, \bar{u})$ is optimal. Consider the time function
$$h \colon t \in [0,T] \mapsto \langle B^\top p(t), \bar{u}(t) \rangle \in \mathbb{R}.$$
A time $t$ is called Lebesgue point if
$$h(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} h(s) \mathrm{d}s.$$
Lebesgue differentiation theorem states that almost every time $t$ is a Lebesgue function, since $h \in L^1(0,T)$.
Let $t$ be a Lebesgue point. Let $v \in U$. Let $u_\varepsilon$ be defined by
$$u_\varepsilon(s) = \begin{cases} v & \text{if } s \in (t-\varepsilon, t+\varepsilon) \\ \bar{u}(s) & \text{otherwise.} \end{cases}$$

**[Proof of Pontryagin's principle]** The same calculation as above leads to:
$$\begin{aligned}
0 &\leq \frac{1}{2\varepsilon} \langle q, y[u_\varepsilon](T) - \bar{y}(T) \rangle \\
&= \frac{1}{2\varepsilon} \int_0^T \langle B^\top p(s), u_\varepsilon(t) - \bar{u}(s) \rangle \, \mathrm{d}s \\
&= \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} \langle B^\top p(s), v - \bar{u}(s) \, \mathrm{d}s \\
&\underset{\varepsilon \downarrow 0}{\longrightarrow} \langle B^\top p(t), v - \bar{u}(t) \rangle,
\end{aligned}$$
as was to be proved.

**[Pontryagin for time-optimal problems]** We come back to the **time-optimal control** problem $(P)$.

*Theorem 16.* (Pontryagin's principle). Let $y_0 \notin C$, assume that $\bar{T} < \infty$. Let $(\bar{y}, \bar{u})$ be a solution to the original minimum time problem $(P)$.
Then, there exists $\bar{q} \in N_C(\bar{y}(\bar{T}))$, $\bar{q} \neq 0$ such that

$$\bar{u}(t) \in \operatorname*{argmin}_{v \in U} H(v, \bar{y}(t), p(t)) = \operatorname*{argmin}_{v \in U} \langle B^\top p, v \rangle, \tag{5}$$

where $p$ is the solution to the costate equation:

$$-\dot{p}(t) = A^\top p(t), \quad p(\bar{T}) = \bar{q}.$$

*Remark.* Pontryagin's principle is only a necessary optimality condition.

**[Proof]**

- By Lemma 13 and by Corollary 14, there exists $\bar{q} \in N_C(\bar{y}(\bar{T}))$ such that
  $$\langle \bar{q}, z \rangle \leq \langle \bar{q}, y \rangle, \quad \forall z \in C, \quad \forall y \in \mathcal{R}(\bar{T}).$$
  We take $z = \bar{y}(\bar{T}) \in C$.
- It follows that $(\bar{y}, \bar{u})$ is a **solution to the auxiliary problem** $(P_{\text{aux}}[q,T])$, with $q = \bar{q}$ and $T = \bar{T}$.
- Applying Pontryagin's principle to the auxiliary problem (Theorem 15), we obtain (5).

## 4. BACK TO THE LUNAR LANDING PROBLEM

**[Lunar landing problem]** Recall the problem:

$$\inf_{\substack{T \geq 0 \\ h\colon [0,T] \to \mathbb{R} \\ v\colon [0,T] \to \mathbb{R} \\ u\colon [0,T] \to \mathbb{R}}} T, \quad \begin{cases} \dot{h}(t) = v(t), & h(0) = h_0,\ h(T) = 0 \\ \dot{v}(t) = u(t), & v(0) = v_0,\ v(T) = 0 \\ u(t) \in [-1, 1]. \end{cases}$$

The dynamics writes:

$$\begin{pmatrix} \dot{h}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

The lunar landing problem is a special case of $(P)$, with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \{0\}.$$

**[Lunar landing problem]** We apply Pontryagin's principle. Let $T$ be the optimal time.

- Costate equation (4) reads:
  $$-\begin{pmatrix} \dot{p}_h(t) \\ \dot{p}_v(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_h(t) \\ p_v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ p_h(t) \end{pmatrix}.$$
- Terminal condition: $(p_h(T), p_v(T)) \in N_C(\bar{h}(T), \bar{v}(T)) = \mathbb{R}^2$ does not bring any information!
- Analytic resolution:
  $$p_h(t) = p_h(T), \quad \dot{p}_v(t) = -p_h(t) = -p_h(T)$$
  and thus
  $$p_v(t) = p_v(T) + p_h(T)(T - t).$$

**[Lunar landing problem]**

- The minimization condition reads:
  $$\bar{u}(t) \in \operatorname*{argmin}_{v \in [-1,1]} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \begin{pmatrix} p_h(t) \\ p_v(t) \end{pmatrix} v = \operatorname*{argmin}_{v \in [-1,1]} p_v(t) v.$$
  It follows that
  $$\begin{cases} \bar{u}(t) = -1 & \text{if } p_v(t) > 0 \\ \bar{u}(t) = 1 & \text{if } p_v(t) < 0 \end{cases} \quad \text{for a.e. } t \in [0, T].$$

**[Lunar landing problem]**
We now **prove the original conjecture**: any optimal control is piecewise constant, with at most two pieces, taking values in $\{-1, 1\}$.

- Case 1: $p_h(T) = 0$. Then $p_v(T) \neq 0$. Therefore
  - either $p_v(t) = p_v(T) < 0 \Longrightarrow \bar{u}(t) = 1$
  - or $p_v(t) = p_v(T) > 0 \Longrightarrow \bar{u}(t) = -1$.
- Case 2: $p_h(T) \neq 0$. Then the map $t \mapsto p_v(T) + p_h(T)(T - t)$ vanishes at exactly one point, say $\tau$.
  - If $\tau \leq 0$ or $\tau \geq T$, then the optimal control is constant, equal to 1 or -1.
  - If $\tau \in (0, T)$, then there is a switch.

**[A summary]** Given a linear time-optimal control problem, the following methodology can be followed to analyze it:

(1) Put the state equation in the form $\dot{y} = Ay + Bu$. Check the **assumptions** state at the beginning of Section 2.
(2) **Existence** of a solution: verify the applicability of Theorem 10.
(3) Derive **optimality conditions** with Theorem 15.
(4) Deduce **structural properties** of optimal controls and trajectories.
(5) Transform the problem into a **geometric** problem.
(6) Solve it!