SOD 311 — Lecture 3: HJB equation and viscosity solutions

Laurent Pfeiffer (Inria and CentraleSupélec, University Paris-Saclay)

[Objectives]

- *Goal:* finding global solutions to optimal control problems (in feedback form), by solving a non-linear PDE.
- *Issues:* characterization of the value function with the Hamilton-Jacobi-Bellman equation.

[Bibliography]

The following references are related to Lecture 3:

- M. Bardi and I. Capuzzo-Dolcetta. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Birkhäuser, 1997.
- F. Bonnans et P. Rouchon, Commande et optimisation de systèmes dynamiques, Editions de l'Ecole Polytechnique, 2005. (Partie 3, Chapitres 3 et 4).

1. INTRODUCTION

[Introduction]

- Our results so far were based on optimality conditions (Pontryagin's principle).
- Now: a different approach, based on **dynamic programming**.
  In some sense, more specific to optimal control.
- The dynamic programming principle is ubiquitous in optimization. A very general concept allowing to "split" some problems into a family of simpler problems.
- The central tool: the **value function** V.
  - Defined as the value of the optimization problem, expressed as a function of the initial state.
  - Characterized as the unique **viscosity** solution of a non-linear partial differential equation (PDE) called **HJB equation**.

[Introduction]

- *Interest:* a **globally optimal** solution to the problem can be derived from V.
- *Limitation:* **curse of dimensionality**.
- *Warning:* focus on a specific class of problems.
  All concepts can be extended, in particular to a stochastic framework (finance), and to other nonlinear PDEs.

[Problem formulation]

Data of the problem and assumptions:

- A parameter $\lambda > 0$.
- A non-empty and compact subset U of $\mathbb{R}^m$.
- A mapping $f\colon (u,y) \in U \times \mathbb{R}^n \to \mathbb{R}^n$, such that
$$\begin{aligned} \|f(u,y)\| &\leq \|f\|_\infty, \\ \|f(u',y') - f(u,y)\| &\leq L_f \|(u',y') - (u,y)\|, \end{aligned}$$
  for all (u,y) and $(u',y') \in U \times \mathbb{R}^n$.
- A mapping $\ell\colon (u,y) \in U \times \mathbb{R}^n \to \mathbb{R}$, such that
$$\begin{aligned} \|\ell(u,y)\| &\leq \|\ell\|_\infty, \\ \|\ell(u',y') - \ell(u,y)\| &\leq L_\ell \|(u',y') - (u,y)\|, \end{aligned}$$
  for all (u,y) and $(u',y') \in U \times \mathbb{R}^n$.

[Problem formulation]

- *Notation:* for any $\tau \in [0,\infty]$, $\mathcal{U}_\tau$ is the set of measurable functions from $(0,\tau)$ to U.
- State equation: for $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_\infty$, there is a unique solution $y[u,x]$ to the ODE
$$\dot{y}(t) = f(u(t), y(t)), \quad y(0) = x,$$
  by the theorem of Picard-Lindelöf (Cauchy-Lipschitz).
- Cost function W, for $u \in \mathcal{U}_\infty$ and $x \in \mathbb{R}^n$:
$$W(u,x) = \int_0^\infty e^{-\lambda t} \ell\big(u(t), y[u,x](t)\big) \,\mathrm{d}t.$$
- Optimal control problem and value function V:
$$V(x) = \inf_{u \in \mathcal{U}_\infty} W(u,x). \qquad (P(x))$$

[Grönwall's lemma]

Lemma 1. (Grönwall's lemma). Let $\alpha > 0$ and let $\beta > 0$. Let $\theta\colon [0,\infty) \to \mathbb{R}$ be a continuous function such that
$$\theta(t) \leq \alpha + \beta \int_0^t \theta(s) \,\mathrm{d}s, \quad \forall t \in [0,\infty).$$
Then, $\theta(t) \leq \alpha e^{\beta t}$, for all $t \in [0,\infty)$.

Corollary 2. Let $u \in \mathcal{U}_\infty$. For all x and $\tilde{x}$, for all $t \geq 0$, it holds:
$$\|y[u,x](t) - y[u,\tilde{x}](t)\| \leq e^{L_f t} \|x - \tilde{x}\|.$$

Proof. Grönwall with $\theta = \|y - \tilde{y}\|$, $\alpha = \|x - \tilde{x}\|$, $\beta = L_f$.

[Dynamic programming principle]

Theorem 3. (Dynamic programming (DP) principle). Let $\tau > 0$. Then for all $x \in \mathbb{R}^n$, abbreviating $y = y[u,x]$,

$$V(x) = \inf_{u \in \mathcal{U}_\tau} \Big(\int_0^\tau e^{-\lambda t} \ell\big(u(t), y(t)\big)\, \mathrm{d}t + e^{-\lambda \tau} V(y(\tau)) \Big). \tag{DPP}$$

Interpretation:

- $V(x)$ is the value function of an optimal control problem on the interval $(0,\tau)$.
- The original integral has been truncated:

$$\int_\tau^\infty e^{-\lambda t} \ell\big(u(t), y(t)\big)\, \mathrm{d}t \quad \rightsquigarrow \quad e^{-\lambda \tau} V(y(\tau)).$$

  The term $e^{-\lambda\tau}V(y(\tau))$ is the "optimal cost from τ to ∞".

[Flow property]

Lemma 4. (Flow property). Let $x \in \mathbb{R}^n$ and let $u \in \mathcal{U}_\infty$. Define:

- $u_1 = u_{|(0,\tau)} \in \mathcal{U}_\tau$
- $u_2 = u_{|(\tau,\infty)} \in L^\infty(\tau,\infty;U)$
- $\tilde{u}_2 \in \mathcal{U}_\infty$, $\tilde{u}_2(t) = u_2(t+\tau)$.

It holds:

$$y[u,x](t) = y\big[\tilde{u}_2, y[u_1,x](\tau)\big](t-\tau),$$

for any $t \geq \tau$.

Remark. After time τ, one can forget u_1 and only remember $y[x,u_1](\tau)$.

[Proof]

Proof of the DP-principle. Let us denote

$$\tilde{V}(x) = \inf_{u \in \mathcal{U}_\tau} \Big(\int_0^\tau e^{-\lambda t} \ell\big(u(t), y(t)\big)\, \mathrm{d}t + e^{-\lambda \tau} V(y(\tau)) \Big).$$

Step 1: $V \geq \tilde{V}$. Let u, u_1, u_2, and $\tilde{u}_2$ be as in Lemma 4.

$$\begin{aligned} W(u,x) &= \int_0^\infty e^{-\lambda t} \ell\big(u(t), y[u,x](t)\big)\, \mathrm{d}t \\ &= \int_0^\tau e^{-\lambda t} \ell\big(u(t), y[u,x](t)\big)\, \mathrm{d}t \\ &\quad + e^{-\lambda\tau} \int_\tau^\infty e^{-\lambda(t-\tau)} \ell\big(u(t), y[u,x](t)\big)\, \mathrm{d}t \\ &= \int_0^\tau e^{-\lambda t} \ell\big(u(t), y[u,x](t)\big)\, \mathrm{d}t \\ &\quad + e^{-\lambda\tau} \int_0^\infty e^{-\lambda s} \ell\big(u(s+\tau), y[u,x](s+\tau)\big)\, \mathrm{d}s. \end{aligned}$$

[Proof]

We further have, for the last integral:

$$\begin{aligned} &\int_0^\infty e^{-\lambda s} \ell\big(u(s+\tau), y[u,x](s+\tau)\big)\, \mathrm{d}s \\ &\quad = \int_0^\infty e^{-\lambda s} \ell\big(\tilde{u}_2(s), y[\tilde{u}_2, y[u_1,x](\tau)](s)\big)\, \mathrm{d}s \\ &\quad = W(\tilde{u}_2, y[u_1,x](\tau)) \geq V(y[u_1,x](\tau)). \end{aligned}$$

Injecting in the above equality:

$$\begin{aligned} W(u,x) &\geq \int_0^\tau e^{-\lambda t} \ell\big(u_1(t), y[u_1,x](t)\big)\, \mathrm{d}t \\ &\qquad + e^{-\lambda\tau} V(y[u_1,x](\tau)) \\ &\geq \tilde{V}(x). \end{aligned}$$

Minimizing with respect to u yields $V \geq \tilde{V}$.

[Proof]

Step 2: $\tilde{V} \leq V$. Let $\varepsilon > 0$. Let $u_1 \in \mathcal{U}_\tau$ be such that

$$\begin{aligned} &\int_0^\tau e^{-\lambda t} \ell\big(u_1(t), y[u_1](t)\big)\, \mathrm{d}t + e^{-\lambda\tau} V(y[u_1,x](\tau)) \\ &\qquad \leq \tilde{V}(x) + \varepsilon/2. \end{aligned}$$

Let $\tilde{u}_2 \in \mathcal{U}_\infty$ be such that

$$W(\tilde{u}_2, y[u_1,x](\tau)) \leq V(y[u_1,x](\tau)) + \varepsilon/2.$$

Let u be defined by

$$u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0,\tau), \\ \tilde{u}_2(t-\tau) & \text{for a.e. } t \in (\tau,\infty). \end{cases}$$

[Proof]

The same calculation as above yields:

$$\begin{aligned} W(u,x) &= \int_0^\tau e^{-\lambda t} \ell\big(u_1(t), y[u_1,x](t)\big)\, \mathrm{d}t \\ &\quad + e^{-\lambda\tau} \underbrace{\int_0^\infty e^{-\lambda t} \ell\big(\tilde{u}_2(t), y[\tilde{u}_2(t), y[u_1,x](\tau)](t)\big)\, \mathrm{d}t}_{=W(\tilde{u}_2, y[u_1,x](\tau)))} \end{aligned}$$

Therefore,

$$\begin{aligned} W(u,x) &\leq \int_0^\tau e^{-\lambda t} \ell\big(u_1(t), y[u_1,x](t)\big)\, \mathrm{d}t \\ &\qquad + e^{-\lambda\tau} \big(V(y[u_1,x](\tau)) + \varepsilon/2\big) \\ &\leq \tilde{V}(x) + \varepsilon. \end{aligned}$$

It follows that

$$V(x) \leq \tilde{V}(x) + \varepsilon, \quad \forall \varepsilon > 0.$$

[Decoupling]

Corollary 5.
- Let $u \in \mathcal{U}_\infty$ be a solution to $P(x)$. Let $\tau > 0$. Let u_1 and $\tilde{u}_2$ be defined as in Lemma 4. Then,
  - u_1 is **optimal in the DP principle**
  - $\tilde{u}_2$ is **optimal** for $P(y[u_1, x](\tau))$.
- Conversely: let u_1 be a minimizer of (DPP). Let $\tilde{u}_2$ be a solution to $P(y[u_1, x])(\tau)$. Let $u \in \mathcal{U}_\infty$ be defined by

$$u(t) = \begin{cases} u_1(t) & \text{for a.e. } t \in (0, \tau) \\ \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). \end{cases}$$

  Then u is a solution to $P(x)$.

What can we do with the value function? If V is known, then the DP-principle allows to **decouple** the problem in time.

3. A FIRST CHARACTERIZATION OF THE VALUE FUNCTION

[Regularity of V]

Lemma 6. The value function V is bounded. It is also uniformly continuous, that is, for all $\varepsilon > 0$, there exists $\alpha > 0$ such that for all x and $\tilde{x} \in \mathbb{R}^n$,

$$\|\tilde{x} - x\| \leq \alpha \Longrightarrow |V(\tilde{x}) - V(x)| \leq \varepsilon.$$

Proof. Step 1: proof of boundedness. Let $x \in \mathbb{R}^n$ and $u \in \mathcal{U}_\infty$. We have

$$|W(x, u)| \leq \int_0^\infty e^{-\lambda t} \|\ell\|_\infty \, \mathrm{d}t \leq \frac{1}{\lambda} \|\ell\|_\infty.$$

Thus $|V(x)| \leq \frac{1}{\lambda} \|\ell\|_\infty$.

[Regularity of V]

Step 2: proof of uniform continuity. Let $\varepsilon > 0$. Let $\alpha > 0$. Let x and $\tilde{x}$ be such that $\|\tilde{x} - x\| \leq \alpha$, we will specify α later. We have:

$$|V(\tilde{x}) - V(x)| = \big| \inf_{u \in \mathcal{U}_\infty} W(\tilde{x}, u) - \inf_{u \in \mathcal{U}_\infty} W(x, u) \big|$$
$$\leq \sup_{u \in \mathcal{U}_\infty} \big| W(\tilde{x}, u) - W(x, u) \big| \leq \Delta_1 + \Delta_2,$$

where

$$\Delta_1 = \sup_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t} \big| \ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t)) \big| \, \mathrm{d}t$$
$$\Delta_2 = \sup_{u \in \mathcal{U}_\infty} \int_\tau^\infty e^{-\lambda t} \big| \ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t)) \big| \, \mathrm{d}t,$$

where $\tilde{y} = y[\tilde{x}, u]$ and $y = y[x, u]$ and where $\tau > 0$ is **arbitrary**.

[Regularity of V]

- Bound of Δ_1. By Corollary 2,
  $\|\tilde{y}(t) - y(t)\| \leq e^{L_f t} \|\tilde{x} - x\| \leq e^{L_f \tau} \alpha, \quad \forall t \in [0, \tau].$
  Therefore, $\Delta_1 \leq \tau L_\ell e^{L_f \tau} \alpha$.
- Bound of Δ_2. Since ℓ is bounded,

$$\Delta_2 \leq 2\|\ell\|_\infty \int_\tau^\infty e^{-\lambda t} \, \mathrm{d}t = \frac{2\|\ell\|_\infty}{\lambda} e^{-\lambda \tau}.$$

Conclusion: take $\tau > 0$ sufficiently large, so that $\Delta_2 \leq \frac{\varepsilon}{2}$.
Take then α sufficiently small, so that $\Delta_1 \leq \frac{\varepsilon}{2}$.
The construction of α is independent of x and $\tilde{x}$.
We have $|V(x) - V(\tilde{x})| \leq \varepsilon$.

[(More) regularity of V]

Lemma 7. We have
- if $\lambda < L_f$, then V is (λ / L_f)-Hölder continuous
- if $\lambda = L_f$, then V is α-Hölder continuous for all $\alpha \in (0, 1)$
- if $\lambda > L_f$, then V is Lipschitz continuous.

Exercise: prove the last statement of the lemma.

[(More) regularity of V]

Proof of the last case. We have

$$|V(\tilde{x}) - V(x)|$$
$$\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| \, \mathrm{d}t$$
$$\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} L_\ell \|\tilde{y}(t) - y(t)\| \, \mathrm{d}t$$
$$\leq \int_0^\infty e^{-\lambda t} L_\ell e^{L_f t} \|\tilde{x} - x\| \, \mathrm{d}t$$
$$\leq \frac{L_\ell}{\lambda - L_f} \|\tilde{x} - x\|.$$

[DP-mapping]

Notation: $BUC(\mathbb{R}^n)$ is the set of **bounded and uniformly continuous** functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Lemma 8. The space $BUC(\mathbb{R}^n)$, equipped with the uniform norm (denoted $\| \cdot \|_\infty$) is a **Banach** space.

Fix $\tau > 0$. Consider the "**DP-mapping**":

$$\mathcal{T} \colon v \in BUC(\mathbb{R}^n) \mapsto \mathcal{T}v \in BUC(\mathbb{R}^n),$$

defined by

$$\mathcal{T}v(x) = \inf_{u \in \mathcal{U}_\tau} \Big(\int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, \mathrm{d}t + e^{-\lambda \tau} v(y(\tau)) \Big),$$

where $y = y[u, x]$. *Exercise:* verify that $\mathcal{T}v \in BUC(\mathbb{R}^n)$.

[DP-mapping]

Proof. Let $v \in BUC(\mathbb{R}^n)$. Let $\varepsilon > 0$. Let $\alpha_0 > 0$ be such that

$$\|\tilde{x} - x\| \leq \alpha_0 \Longrightarrow |v(\tilde{x}) - v(x)| \leq \varepsilon/2.$$

Let $\alpha > 0$. Let x and $\tilde{x} \in \mathbb{R}^n$ be such that $\|\tilde{x} - x\| \leq \alpha$.
The value of α will be fixed later.
For all $u \in U_\tau$, for all $t \in [0, \tau]$, we have

$$\|y[u, \tilde{x}](t) - y[u, x](t)\| \leq e^{L_f t} \|\tilde{x} - x\| \leq e^{L_f \tau} \alpha.$$

We have $|\mathcal{T}v(\tilde{x}) - \mathcal{T}v(x)| \leq \Delta_1 + \Delta_2$, with...

[DP-mapping]

$$\Delta_1 = \sup_{u \in \mathcal{U}_\tau} \Big| \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, \mathrm{d}t - \int_0^\tau e^{-\lambda t} \ell(u(t), \tilde{y}(t)) \, \mathrm{d}t \Big|,$$

$$\Delta_2 = \sup_{u \in \mathcal{U}_\tau} \big| e^{-\lambda \tau} v(\tilde{y}(\tau)) - e^{-\lambda \tau} v(y(\tau)) \big|.$$

We fix now

$$\alpha = e^{-L_f \tau} \min \Big(\alpha_0, \frac{\varepsilon}{2\tau} \Big).$$

We have

$$\Delta_1 \leq \tau L_\ell e^{\tau L_f} \alpha \leq \varepsilon/2 \quad \text{and} \quad \Delta_2 \leq \varepsilon/2,$$

since $\|\tilde{y}(\tau) - y(\tau)\| \leq e^{L_f \tau} \alpha \leq \alpha_0$. Therefore,

$$|\mathcal{T} v(\tilde{x}) - \mathcal{T} v(x)| \leq \varepsilon.$$

[DP-mapping]

Lemma 9. The operator $\mathcal{T}$ is **Lipschitz continuous** with modulus $e^{-\lambda \tau}$.

Proof. Let $x \in \mathbb{R}^n$. We have

$$\begin{aligned} &|\mathcal{T} \tilde{v}(x) - \mathcal{T} v(x)| \leq \\ &\quad \leq \sup_{u \in \mathcal{U}_\tau} \big| e^{-\lambda \tau} \tilde{v}(y[x,u](\tau)) - e^{-\lambda \tau} v(y[x,u](\tau)) \big| \\ &\quad \leq e^{-\lambda \tau} \|\tilde{v} - v\|_\infty. \end{aligned}$$

We conclude that

$$\|\mathcal{T} \tilde{v} - \mathcal{T} v\|_\infty \leq e^{-\lambda \tau} \|\tilde{v} - v\|_\infty.$$

[A characterization of V]

Lemma 10. The value function V is the **unique solution** of the fixed-point equation:

$$\mathcal{T} v = v, \quad v \in BUC(\mathbb{R}^n).$$

Proof.

- Existence: direct consequence of the DP principle ($V = \mathcal{T} V$).
- Uniqueness: for any v such that $v = \mathcal{T} v$, we have
  $$\|v - V\|_\infty = \|\mathcal{T} v - \mathcal{T} V\|_\infty \leq e^{-\lambda \tau} \|v - V\|_\infty.$$
  Thus $v = V$.

Remark: the dynamic programming principle entirely characterises the value function!

[Min-plus linearity]

Notation. Given v_1 and $v_2 \in BUC(\mathbb{R}^n)$, we write $v_1 \leq v_2$ if $v_1(x) \leq v_2(x)$ for all $x \in \mathbb{R}^n$. We define $\min(v_1, v_2) \in BUC(\mathbb{R}^n)$ by

$$\min(v_1, v_2)(x) = \min(v_1(x), v_2(x)), \quad \forall x \in \mathbb{R}^n.$$

Given $\alpha \in \mathbb{R}$, we define $v_1 + \alpha$ by $(v_1 + \alpha)(x) = v_1(x) + \alpha$.

Lemma 11. Let v_1 and $v_2 \in BUC(\mathbb{R}^n)$. Let $\alpha \in \mathbb{R}$. The map $\mathcal{T}$ is monotone:

$$v_1 \leq v_2 \Longrightarrow \mathcal{T} v_1 \leq \mathcal{T} v_2$$

and min-plus linear:

$$\begin{aligned} \min(\mathcal{T} v_1, \mathcal{T} v_2) &= \mathcal{T} \min(v_1, v_2), \\ \mathcal{T}(v + \alpha) &= (\mathcal{T} v) + e^{-\lambda \tau} \alpha. \end{aligned}$$

Proof: exercise.

4. HJB EQUATION: THE CLASSICAL SENSE

[Hamiltonian

We define the **pre-Hamiltonian** H and the **Hamiltonian** $\mathcal{H}$ by

$$\begin{aligned} H(u,x,p) &= \ell(u,x) + \langle p, f(u,x) \rangle, \\ \mathcal{H}(x,p) &= \min_{u \in U} \; H(u,x,p). \end{aligned}$$

Lemma 12. The mapping $\mathcal{H}$ is **continuous**, **concave** with respect to p, and **Lipschitz continuous** with respect to p with modulus $\|f\|_\infty$.

Proof. The pre-Hamiltonian H is affine in p, thus concave in p. As an infimum of concave functions, $\mathcal{H}$ is concave. We have:

$$\begin{aligned} |\mathcal{H}(x,\tilde{p}) - \mathcal{H}(x,p)| &\leq \sup_{u \in U} \; |H(u,x,\tilde{p}) - H(u,x,p)| \\ &\leq \sup_{u \in U} \; |\langle \tilde{p} - p, f(u,x) \rangle| \leq \; \|\tilde{p} - p\| \cdot \|f\|_\infty. \end{aligned}$$

[Informal derivation]

Notation: $C^1(\mathbb{R}^n)$, the set of continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$.

Lemma 13. Let $\Phi \in C^1(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$, let $u \in \mathcal{U}_\infty$, let $y = y[u,x]$. Consider the mapping:

$$\varphi \colon \tau \in [0,\infty) \mapsto \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, \mathrm{d}t + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x).$$

Then $\varphi(0) = 0$ and $\varphi \in W^{1,\infty}(0,\infty)$ with

$$\dot{\varphi}(\tau) = e^{-\lambda \tau} \big(H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau)) \big). \tag{$*$}$$

In particular: $\dot{\varphi}(0) = H(u(0), x, \nabla \Phi(x)) - \lambda \Phi(x)$ (if u is continuous at 0).

[Informal derivation]

Proof. To simplify, we only consider the case where u is continuous, so that y is C^1 and φ is $C^1(\mathbb{R}^n)$. We have then:

$$\begin{aligned} \dot{\varphi}(\tau) &= e^{-\lambda \tau} \ell(u(\tau), y(\tau)) + e^{-\lambda \tau} \langle \nabla \Phi(y(\tau)), \dot{y}(\tau) \rangle \\ &\qquad - \lambda e^{-\lambda \tau} \Phi(y(\tau)) \\ &= e^{-\lambda \tau} \big[\ell(u(\tau), y(\tau)) + \langle \nabla \Phi(y(\tau)), f(u(\tau), y(\tau)) \rangle \big] \\ &\qquad - \lambda e^{-\lambda \tau} \Phi(y(\tau)) \\ &= e^{-\lambda \tau} \big[H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau)) \big]. \end{aligned}$$

[HJB in the classical sense]

Theorem 14. Let $x \in \mathbb{R}^n$. Assume that

- V is continuously differentiable in a neighborhood of x
- $P(x)$ has a solution $\bar{u}$ which is continuous at time 0.

Then,

$$\begin{aligned} &\lambda V(x) - \mathcal{H}(x, \nabla V(x)) = 0, \\ &\bar{u}(0) \in \operatorname*{argmin}_{u_0 \in U} \; H(u_0, x, \nabla V(x)). \end{aligned}$$

[HJB in the classical sense]
Proof. Step 1. Let $u_0 \in U$, let u be the constant control equal to u_0, let $y = y[u, x]$. By the **dynamic programming** principle, we have:

$$0 \le \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(u(t), y(t))\,\mathrm{d}t + e^{-\lambda\tau} V(y(\tau)) - V(x),$$

for all τ. Since $\varphi(0) = 0$, we deduce from $(*)$ that:

$$0 \le \dot{\varphi}(0) = H(u_0, x, \nabla V(x)) - \lambda V(x).$$

Therefore,

$$0 \le H(u_0, x, \nabla V(x)) - \lambda V(x), \quad \forall u_0 \in U.$$

[HJB in the classical sense]
Step 2. Let us apply the **dynamic programming principle** again. Redefining φ and setting $\bar{y} = y[\bar{u}, x]$, we obtain:

$$0 = \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t))\,\mathrm{d}t + e^{-\lambda\tau} V(\bar{y}(\tau)) - V(x),$$

for all $\tau \ge 0$. It follows that

$$0 = H(\bar{u}(0), x, \nabla V(x)) - \lambda V(x).$$

Step 3. It follows that for all $u_0 \in U$,

$$H(\bar{u}(0), x, \nabla V(x)) = \lambda V(x) \le H(u_0, x, \nabla V(x)).$$

Therefore, $H(\bar{u}(0), x, \nabla V(x)) = \mathcal{H}(x, \nabla V(x))$.

[HJB in the classical sense]
Corollary 15. Let $t \ge 0$, assume that $\bar{u}$ is continuous in a neighborhood of t and that V is C^1 in a neighborhood of $\bar{y}(t)$, where $\bar{y} := y[\bar{u}, x](t)$. Then,

$$\bar{u}(t) \in \operatorname*{argmin}_{u_0 \in U} H(u_0, \bar{y}, \nabla V(\bar{y})).$$

[HJB in the classical sense]
Remarks.
Let us define the Q-function by $Q(u, y) := H(u, y, \nabla V(y))$, assuming that $V \in C^1(\mathbb{R}^n)$.

- If the minimizer is unique in the following relation, we have a **feedback law**:
$$\bar{u}(t) = \operatorname*{argmin}_{U} Q(\cdot, \bar{y}(t)).$$
- In some cases, one can show that $\nabla V(\bar{y}(t)) = p(t)$, where p is defined by some adjoint equation $\to$ **Pontryagin's principle**.
- In **Reinforcement Learning**, the approximation of Q is a central objective.

We will call the equation

$$\lambda v(x) - \mathcal{H}(x, \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n \qquad \text{(HJB)}$$

Hamilton-Jacobi-Bellman equation, with unknown $v \colon \mathbb{R}^n \to \mathbb{R}$.
Remarks.

- In general V is not differentiable $\to$ in **which sense** is the HJB equation to be understood?
- In Theorem 14, we have shown that
$$\bar{u}(t) \in \operatorname{argmin} H(u_0, y[\bar{u}(t), x], \nabla V(y[\bar{u}(t), x])),$$
(under restrictive assumptions). We will see next that this necessary condition is also **sufficient**.

Theorem 16. (Verification). Let us assume the assumptions of Theorem 14 hold for all $x \in \mathbb{R}^n$, so that the **HJB equation is satisfied in the classical sense**. Let $x \in \mathbb{R}^n$. Assume that there exists a control $\bar{u}$ such that

$$\bar{u}(t) \in \operatorname*{argmin}_{u_0 \in U} H(u_0, \bar{y}(t), \nabla V(\bar{y}(t))),$$

where $\bar{y} = y[\bar{u}, x]$. Then $\bar{u}$ is **globally optimal**.

Proof. Consider the function:

$$\varphi(\tau) = \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t))\,\mathrm{d}t + e^{-\lambda\tau} V(\bar{y}(\tau)) - V(x).$$

We have $\varphi(0) = 0$. Using $(*)$ and Theorem 14, we obtain:

$$\begin{aligned}\dot{\varphi}(\tau) &= e^{-\lambda\tau}\big[H(\bar{u}(\tau), \bar{y}(\tau), \nabla V(\bar{y}(\tau)) - V(\bar{y}(\tau))\big] \\ &= e^{-\lambda\tau}\big[\mathcal{H}(\bar{y}(\tau), \nabla V(\bar{y}(\tau))) - V(\bar{y}(\tau))\big] \\ &= 0.\end{aligned}$$

Thus φ is constant, equal to 0. Its limit is given by:

$$0 = \int_0^\infty e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t))\,\mathrm{d}t - V(x) = W(x, \bar{u}) - V(x),$$

proving the optimality of $\bar{u}$.

5. HJB EQUATION: VISCOSITY SOLUTIONS

[Abstract PDE]
We consider an abstract PDE of the form:

$$\mathcal{F}(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n,$$

where $\mathcal{F} \colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous.
It contains the HJB equation with

$$\mathcal{F}(x, v, p) = \lambda v - \mathcal{H}(x, p).$$

Goal of the section: showing that V is a viscosity solution to the HJB equation.

[Sub- and super-differentials]
Definition 17. Let $v \colon \mathbb{R}^n \to \mathbb{R}$. The following sets are called **sub- and superdifferential**, respectively:

$$D^- v(x) = \Big\{ p \in \mathbb{R}^n \mid \liminf_{y \to x} \frac{v(y) - v(x) - \langle p, y - x\rangle}{\|y - x\|} \ge 0 \Big\}$$

$$D^+ v(x) = \Big\{ p \in \mathbb{R}^n \mid \limsup_{y \to x} \frac{v(y) - v(x) - \langle p, y - x\rangle}{\|y - x\|} \le 0 \Big\}.$$

Exercise. Let $v(x) = |x|$. Show that $D^- v(0) = [-1, 1]$.

[Sub- and super-differentials]
We have the following characterization.

Lemma 18. Let $v\colon \mathbb{R}^n \to \mathbb{R}$ be continuous. Let $p \in \mathbb{R}^n$.

- $p \in D^-v(x) \iff$ there exists $\Phi \in C^1(\mathbb{R}^n)$ such that $\nabla\Phi(x) = p$ and $v - \Phi$ has a local minimum in x.
- $p \in D^+v(x) \iff$ there exists $\Phi \in C^1(\mathbb{R}^n)$ such that $\nabla\Phi(x) = p$ and $v - \Phi$ has a local maximum in x.

Proof. The implication $\Longrightarrow$ is admitted. The implication $\Longleftarrow$ is left as an exercise.

[Sub- and super-differentials]
Remark. In the above lemma, one can chose $\Phi(x) = v(x)$ without loss of generality. Thus, we have:

- $(v - \Phi)$ has a local minimum in $x \iff v - \Phi$ is nonnegative in a neighborhood of $x \iff v$ is locally bounded from below by Φ
- $(v - \Phi)$ has a local maximum in $x \iff v - \Phi$ is nonpositive in a neighborhood of $x \iff v$ is locally bounded from above by Φ

Remark. If v is Fréchet differentiable at x, then the sub- and superdifferential are equal to $\{\nabla v(x)\}$.

[Viscosity solutions]
Definition 19. Let $v\colon \mathbb{R}^n \to \mathbb{R}$. We call v a **viscosity subsolution** if
$$\mathcal{F}(x, v(x), p) \leq 0, \quad \forall x \in \mathbb{R}^n,\ \forall p \in D^+v(x)$$
or, equivalently, if for all $\Phi \in C^1(\mathbb{R}^n)$ such that $v - \Phi$ has a local maximum in x,
$$\mathcal{F}(x, v(x), \nabla\Phi(x)) \leq 0.$$

[Viscosity solutions]
Definition 20. Let $v\colon \mathbb{R}^n \to \mathbb{R}$. We call v a **viscosity supersolution** if
$$\mathcal{F}(x, v(x), p) \geq 0, \quad \forall p \in D^-v(x)$$
or, equivalently, if for all $\Phi \in C^1(\mathbb{R}^n)$ such that $v - \Phi$ has a local minimum in x,
$$\mathcal{F}(x, v(x), \nabla\Phi(x)) \geq 0.$$
We call v a **viscosity solution** if it is a sub- and a supersolution.

[Viscosity solutions]
Theorem 21. The value function V is a **viscosity solution** of the HJB equation.

Step 1: V is a subsolution. Let $x \in \mathbb{R}^n$, let $\Phi \in C^1(\mathbb{R}^n)$ be such that $V - \Phi$ has a local maximizer in x and $V(x) = \Phi(x)$.

We have to prove that
$$\lambda v(x) - \mathcal{H}(x, \nabla\Phi(x)) \leq 0.$$
Let $u_0 \in U$, let u be the constant control equal to u_0 and let $y = y[u,x]$. By the DPP, we have:
$$V(x) \leq \int_0^\tau e^{-\lambda t}\ell(u_0, y(t))\,\mathrm{d}t + e^{-\lambda\tau}V(y(\tau)).$$
If τ is sufficiently small, we have $V(y(\tau)) \leq \Phi(y(\tau))$.

[Viscosity solutions]
This implies that for τ sufficiently small,
$$0 \leq \int_0^\tau e^{-\lambda t}\ell(u_0, y(t))\,\mathrm{d}t + e^{-\lambda\tau}\Phi(y(\tau)) - \Phi(x) =: \varphi(\tau).$$
Since $\varphi(0) = 0$, we deduce with $(*)$ that
$$0 \leq \dot{\varphi}(0) = H(u_0, x, \nabla\Phi(x)) - \lambda V(x).$$
Minimizing with respect to $u_0 \in U$, we obtain:
$$0 \leq \mathcal{H}(x, \nabla\Phi(x)) - \lambda V(x),$$
as was to be proved.

[Viscosity solutions]
Step 2: V is supersolution. Let $x \in \mathbb{R}^n$, let $\Phi \in C^1(\mathbb{R}^n)$ be such that $V - \Phi$ has a local minimizer in x and such that $V(x) = \Phi(x)$.
We have to prove that
$$\lambda V(x) - \mathcal{H}(x, \nabla\Phi(x)) \leq 0.$$
It follows from the dynamic programming principle that for $\tau > 0$ small enough
$$\Phi(x) \geq \inf_{u \in \mathcal{U}_\tau} \underbrace{\int_0^\tau e^{-\lambda t}\ell(u(t), y[x,u](t))\,\mathrm{d}t + e^{-\lambda t}\Phi(y[x,u](\tau))}_{=:\varphi[u](\tau)}.$$

[Viscosity solutions]
Thus by Lemma 13,
$$\begin{aligned} 0 &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \dot{\varphi}[u](t)\,\mathrm{d}t \\ &= \inf_{u \in \mathcal{U}_\infty} \int_0^\tau e^{-\lambda t}\big(H(u(t), y[u](t), \nabla\Phi(y[u](t)) - \lambda\Phi(y[u](t))\big)\,\mathrm{d}t \\ &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \underbrace{e^{-\lambda t}\big(\mathcal{H}(y[u](t), \nabla\Phi(y[u](t)) - \lambda\Phi(y[u](t))\big)}_{=:\psi[u](t)}\,\mathrm{d}t. \end{aligned}$$
We have $\psi[u](0) = \mathcal{H}(x, \nabla\Phi(x)) - \lambda V(x)$, in particular, $\psi[u](0)$ does not depend on u.

[Viscosity solutions]
Let $\varepsilon > 0$. There exists (exercise!) $\tau > 0$ such that
$$|\psi[u](t) - \psi[u](0)| \leq \varepsilon, \quad \forall t \in [0,\tau],\ \forall u \in \mathcal{U}_\infty.$$
The previous inequality yields
$$\begin{aligned} 0 &\geq \inf_{u \in \mathcal{U}_\infty} \int_0^\tau \big(\psi[u](0) - \varepsilon\big)\,\mathrm{d}t \\ &\geq \tau(\mathcal{H}(x, \nabla\Phi(x)) - \lambda V(x) - \varepsilon). \end{aligned}$$
Dividing by τ and sending ε to 0, we get the result.

[Viscosity solutions]
Theorem 22. (Comparison principle). Let v_1 be a subsolution to the HJB equation. Let v_2 be a supersolution to the HJB equation. Then
$$v_1(x) \leq v_2(x), \quad \forall x \in \mathbb{R}^n.$$
Proof: admitted.

Corollary 23. The value function V is the **unique** viscosity solution.

Proof. By the comparison principle, any viscosity solution v is such that $v \leq V$ and $v \geq V$.