Optimal Control of Ordinary Differential Equations
SOD 311

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General information

- Pre-requisites: Basic knowledge in integration, calculus and optimization (Lagrange multipliers).
- No programming session, the lectures will include (theoretical) application exercises.
- All lectures take place at Ensta-Paris.
- Two written exams (lecture and personal notes allowed).
- Website: `https://laurentpfeiffer.github.io/sod/`

First part of the course

In the first part of the course, you will learn...

- how to **formulate** an optimal control problem
- how to prove the **existence** of a solution
- how to characterize the solution through **optimality conditions**.

Schedule:

13.09	13:30-17:15	RB	Examples and analytical framework
20.09	13:30-17:15	LP	Time-optimal problems
21.09	8:30-12:15	LP	Linear-quadratic problems
28.09	8:30-12:15	LP/RB	Pontryagin's principle
05.10	8:30-12:15	RB	Pontryagin's principle
12.10	10:00-12:15		**Intermediate written exam**

Second part of the course

In the second part of the course, you will learn to solve **numerically** an optimal control problem...

- with methods deriving from **Pontryagin**'s principle
- with methods deriving from the **dynamic programming** principle.

Schedule:

19.10	8:30-12:15	LP	Model predictive control
26.10	8:30-12:15	RB	Shooting and direct methods
09.11	8:30-12:15	LP	HJB approach
16.11	8:30-10:30	LP	HJB approach
16.11	10:45-12:15		**Final written exam**

Optimal control in a nutshell

A simple optimal control problem:

$$\inf_{\substack{y \in W^{1,\infty}(0,T) \\ u \in L^\infty(0,T)}} \phi(y(T)), \quad \text{subject to: } \begin{cases} \dot{y}(t) = & f(y(t), u(t)), \\ y(0) = & y_0. \end{cases}$$

Vocabulary. Optimization variables:

- u: the control
- y: the state.

Here, the control is said to be **open-loop**, it is a function of time $\rightarrow$ a sequence of pre-defined actions to be executed.

In this lecture

Guideline. We aim at finding an optimal control $\bar{u}$ with associated trajectory $\bar{y}$ in **closed-loop** form:

$$\bar{u}(t) = \kappa(t, \bar{y}(t)), \quad \forall t.$$

The map κ should be independent of the initial condition y_0.

Motivation.

- In some situations: easier to find κ!
- Robustness, flexibility.

Intention.

- Specific techniques from optimal control.
- Overview of the diversity of techniques.

In this lecture

Outline.

- Lecture 1: time-optimal linear problems
- Lecture 2: linear-quadratic problems
- Lecture 3: exercises
- Lectures 4 and 5: HJB equation.

Lecture 1: Time-optimal linear problems

- *Goal:* controlling a dynamical system so as to **reach a target as fast as possible**.
- *Focus:* linear systems $\dot{y}(t) = Ay(t) + Bu(t)$.
- *Issues:* existence of a solution, optimality conditions, graph of feedback κ.

Bibliography

The following references are related to Chapter 1:

- F. Bonnans et P. Rouchon, Commande et optimisation de systèmes dynamiques, Editions de l'Ecole Polytechnique, 2005. (Partie 3, Chapitre 1).
- E. Trélat, Contrôle optimal : théorie et applications. Version électronique, 2013. (Chapitre 3).
- E. Sontag, Mathematical Control Theory, Springer, 1998. (Chapter 10).

Model

A spatial engine has the dynamics:

$$m\ddot{h}(t) = u(t), \quad \forall t \geq 0, \tag{1}$$

where:

m	mass of the engine
$h(t)$	heigth of the engine at time t
$u(t)$	propulsion force at time t
$v(t) = \dot{h}(t)$	velocity at time t.

Problem: given h_0 and v_0, find the smallest $T > 0$ for which there exist time functions h and u satisfying (1),

$$(h(0), v(0)) = (h_0, v_0), \quad \text{and} \quad (h(T), v(T)) = (0, 0).$$

Mathematical problem

For simplicity, we take $m = 1$. We consider constraints on u. Given (h_0, v_0), the problem writes:

$$\inf_{\substack{T \geq 0 \\ h:\,[0,T]\to\mathbb{R} \\ v:\,[0,T]\to\mathbb{R} \\ u:\,[0,T]\to\mathbb{R}}} T, \quad \text{s.t.:} \begin{cases} \dot{h}(t) = v(t), & h(0) = h_0, \quad h(T) = 0, \\ \dot{v}(t) = u(t), & v(0) = v_0, \quad v(T) = 0, \\ u(t) \in [-1, 1]. \end{cases}$$

Remark. The state (h, v) is uniquely defined by the control u (via the dynamical system).

For the moment: no theoretical tool at hand... let's see what we can do!

Mathematical problem

For simplicity, we take $m = 1$. We consider constraints on u. Given (h_0, v_0), the problem writes:

$$\inf_{\substack{T \geq 0 \\ h\colon [0,T] \to \mathbb{R} \\ v\colon [0,T] \to \mathbb{R} \\ u\colon [0,T] \to \mathbb{R}}} T, \quad \text{s.t.:} \begin{cases} \dot{h}(t) = v(t), & h(0) = h_0, \quad h(T) = 0, \\ \dot{v}(t) = u(t), & v(0) = v_0, \quad v(T) = 0, \\ u(t) \in [-1, 1]. \end{cases}$$

Remark. The state (h, v) is uniquely defined by the control u (via the dynamical system).

For the moment: no theoretical tool at hand... let's see what we can do!

Accelerating trajectories

For $u = 1$, we have

$$\begin{cases} v(t) = & v_0 + t \\ h(t) = & h_0 + tv_0 + \frac{1}{2}t^2. \end{cases}$$

We can isolate t in the first line: $t = v(t) - v_0$ and inject the result in the second line:

$$h(t) = h_0 + (v(t) - v_0)v_0 + \frac{1}{2}(v(t) - v_0)^2.$$

The curve

$$\{(h(t), v(t)) \mid t \geq 0\}$$

is the portion of a **parabola**.

Accelerating trajectories

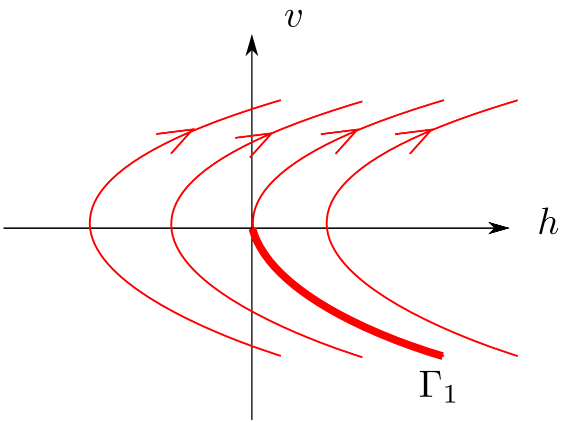


Figure: Trajectories for $u = 1$ (acceleration).

Accelerating trajectories

Let Γ_1 denote the set of initial conditions for which $u = 1$ steers (h, v) to $(0, 0)$. We have:

$$(h_0, v_0) \in \Gamma_1 \iff \begin{cases} \exists T \geq 0 \\ 0 = v_0 + T \\ 0 = h_0 + T v_0 + \frac{1}{2} T^2 \end{cases} \iff \begin{cases} v_0 \leq 0 \\ 0 = h_0 - v_0^2 + \frac{1}{2} v_0^2. \end{cases}$$

Therefore,

$$\Gamma_1 = \left\{ (h_0, v_0) \in \mathbb{R}^2 \;\middle|\; \begin{array}{rl} v_0 \leq & 0 \\ h_0 = & \frac{1}{2} v_0^2. \end{array} \right\}.$$

Decelerating trajectories

For $u = -1$, we have

$$\begin{cases} v(t) = & v_0 - t \\ h(t) = & h_0 + tv_0 - \frac{1}{2}t^2. \end{cases}$$

We can isolate t in the first line: $t = v_0 - v(t)$ and inject the result in the second line:

$$h(t) = h_0 + (v_0 - v(t))v_0 - \frac{1}{2}(v_0 - v(t))^2.$$

The curve

$$\{(h(t), v(t)) \,|\, t \geq 0\}$$

is the portion of a **parabola**.

Decelerating trajectories

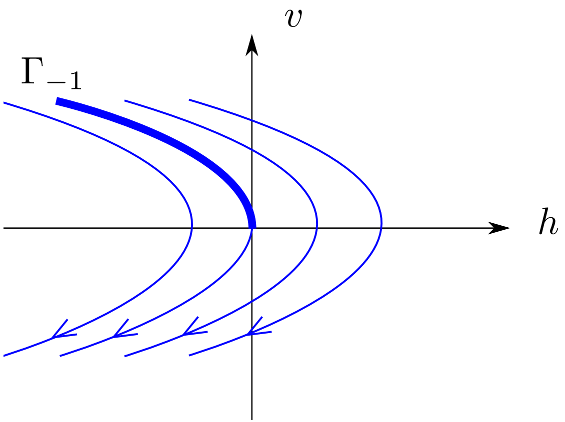


Figure: Trajectories for $u = -1$ (deceleration).

Decelerating trajectories

Let Γ_{-1} denote the set of initial conditions for which $u = -1$ steers (h, v) to $(0, 0)$. We have:

$$(h_0, v_0) \in \Gamma_{-1} \iff \begin{cases} \exists T \geq 0 \\ 0 = v_0 - T \\ 0 = h_0 + Tv_0 - \frac{1}{2}T^2 \end{cases} \iff \begin{cases} v_0 \geq 0 \\ 0 = h_0 + v_0^2 - \frac{1}{2}v_0^2. \end{cases}$$

Therefore,

$$\Gamma_{-1} = \left\{ (h_0, v_0) \in \mathbb{R}^2 \;\middle|\; \begin{array}{rl} v_0 \geq & 0 \\ h_0 = & -\frac{1}{2}v_0^2 \end{array} \right\}.$$

A simple case

Consider the case $v_0 = 0$.

Then we should (fully) accelerate and (fully) decelerate on equal intervals of time.

- If $h_0 < 0$: accelerate ($u = 1$) until $h(t) = h_0/2$, then decelerate ($u = -1$).
- If $h_0 > 0$: decelerate ($u = -1$) until $h(t) = h_0/2$, then accelerate ($u = 1$).

A simple case

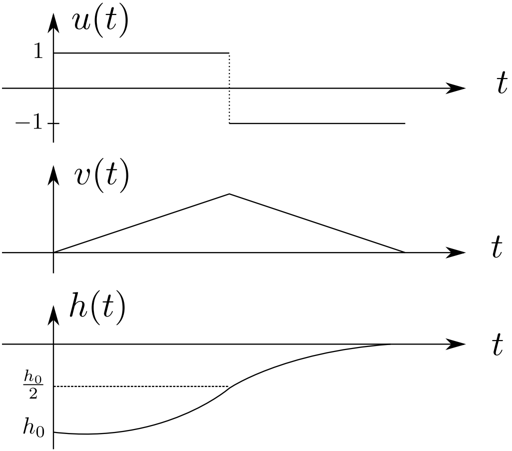


Figure: Optimal control and trajectory for $v_0 = 0$ and $h_0 < 0$.

A simple case

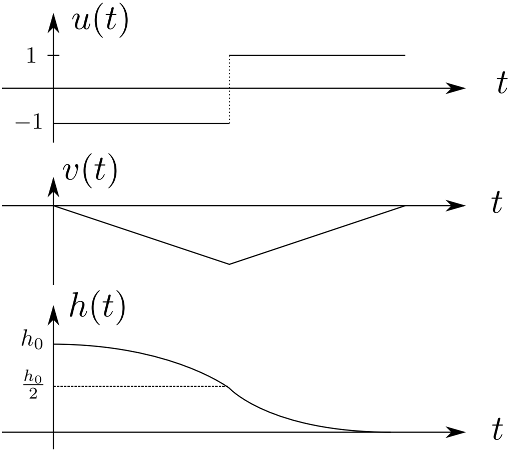


Figure: Optimal control and trajectory for $v_0 = 0$ and $h_0 > 0$.

A simple case

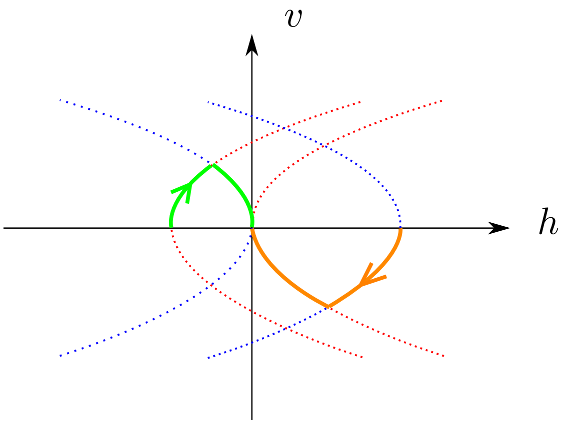


Figure: Some optimal trajectories with null initial speed.

General case

The theory (developed in the next sections) tells us the following.

For any $(h_0, v_0) \in \mathbb{R}^2$,

- There exists an optimal time $\bar{T}$ and an optimal control $\bar{u}$.
- Any optimal control takes values in $\{-1, 1\}$.
- Any optimal control is **piecewise constant**, with **atmost two pieces**.

General case

In other words, for any optimal control $\bar{u}$, one of the following cases is satisfied:

1. $\bar{u}(t) = 1$, for almost every $t \in (0, \bar{T})$
2. $\bar{u}(t) = -1$, for a.e. $t \in (0, \bar{T})$
3. "Accelerate-Decelerate": $\exists \tau \in (0, \bar{T})$ such that:
   $\bar{u}(t) = 1$, for a.e. $t \in (0, \tau)$, $\bar{u}(t) = -1$, for a.e. $t \in (\tau, \bar{T})$.
4. "Decelerate-Accelerate": $\exists \tau \in (0, \bar{T})$ such that:
   $\bar{u}(t) = -1$, for a.e. $t \in (0, \tau)$, $\bar{u}(t) = 1$, for a.e. $t \in (\tau, \bar{T})$.

In the last two cases, τ is called **switching time**.

Remark for French readers: we use the english notation (a, b) for the open interval, instead of the french notation $]a, b[$.

General case

The problem is reduced to a **geometric** problem.

Find all trajectories such that...

- starting at the initial condition,
- ending up at the origin,
- made of two portions of parabola (a “red” and a “blue” one).

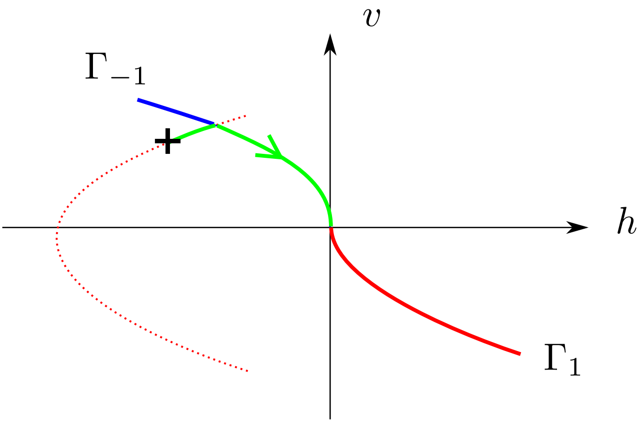
We will call them **Pontryagin** trajectories.

Methodology: for each initial condition,

- find all possible Pontryagin trajectories,
- find out the optimal one (there may exist Pontryagin trajectories which are not optimal).

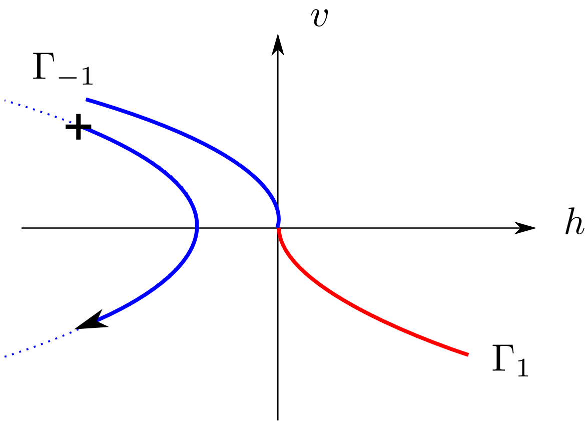
General case

First case: (h_0, v_0) lies strictly under $\Gamma_1 \cup \Gamma_{-1}$.
One possibility for the scenario "accelerate-decelerate".



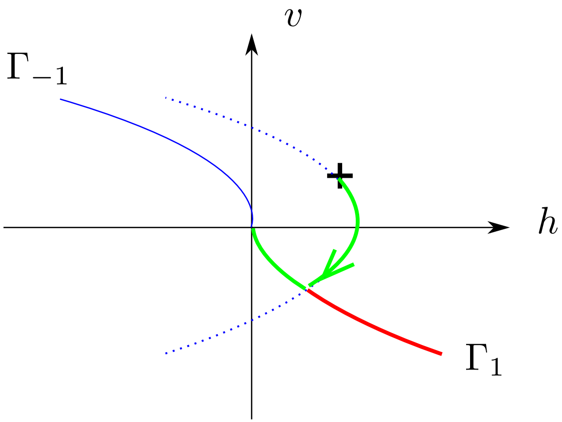
General case

First case: (h_0, v_0) lies strictly under $\Gamma_1 \cup \Gamma_{-1}$.
Zero possibility for the scenario "decelerate-accelerate".



General case

Second case: (h_0, v_0) lies strictly above $\Gamma_1 \cup \Gamma_{-1}$.
One possibility for the scenario "decelerate-accelerate".



General case

Second case: (h_0, v_0) lies strictly above $\Gamma_1 \cup \Gamma_{-1}$.
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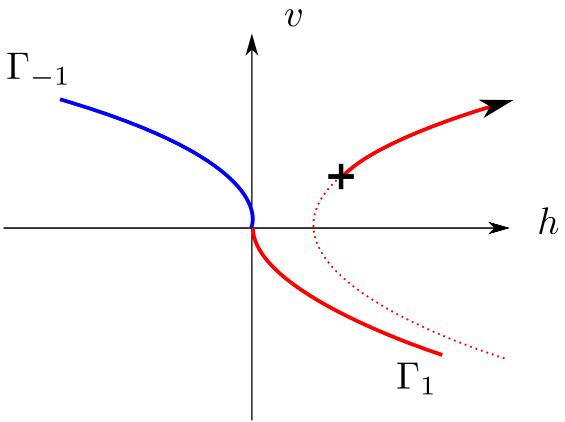

General case

Conclusion: Whatever the initial condition, there is exactly one Pontryagin trajectory, which is necessarily optimal.

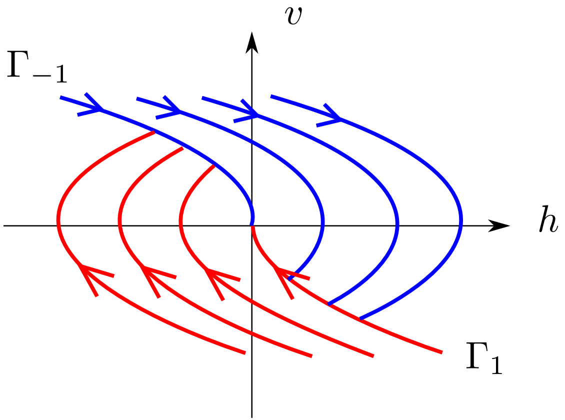

Figure: Phase portrait of optimal trajectories.

General case

We finally obtain a relation in **feedback form** for optimal controls $\bar{u}$ with associated trajectory $(\bar{h}, \bar{v})$:

$$\bar{u}(t) = \kappa(\bar{h}(t), \bar{v}(t)),$$

where κ is defined by:

$$\kappa(h, v) = \begin{cases} 1 & \text{if } (h, v) \in \Gamma_1 \\ -1 & \text{if } (h, v) \in \Gamma_{-1} \\ 1 & \text{if } (h, v) \text{ lies strictly under } \Gamma_{-1} \cup \Gamma_1 \\ -1 & \text{if } (h, v) \text{ lies strictly above } \Gamma_{-1} \cup \Gamma_1, \end{cases}$$

for any $(h, v) \in \mathbb{R}^2 \backslash \{0\}$.

Remark: The feedback relation holds whatever the initial condition of the problem.

Summary

The three main steps of our methodology:

- Calculation of **trajectories with constant controls** (with extremal values).
- Theory $\rightarrow$ **structural properties** of optimal controls.
- Reformulation of the problem as a **geometric** problem.

Framework

A general linear time-optimal control problem:

$$\inf_{\substack{T \geq 0 \\ y \in W^{1,\infty}(0,T;\mathbb{R}^n) \\ u \in L^\infty(0,T;\mathbb{R}^m)}} T, \quad \text{s.t.:} \begin{cases} \dot{y}(t) = & Ay(t) + Bu(t), \\ y(0) = & y_0, \\ y(T) \in & C, \\ u(t) \in & U. \end{cases} \qquad (P)$$

Data of the problem and assumptions:

- Initial condition: $y_0 \in \mathbb{R}^n$
- Dynamics' coefficients: $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$
- A control set: $U \subset \mathbb{R}^m$, assumed convex, compact, non-empty
- A target: $C \subset \mathbb{R}^n$, assumed convex, closed, non-empty.

Matrix exponential

Definition 1

Let $M \in \mathbb{R}^{n\times n}$. We call **matrix exponential** e^M the matrix

$$e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k \in \mathbb{R}^{n\times n}.$$

Lemma 2

- *For any operator norm* $\|\cdot\|$, *we have* $\|e^M\| \leq e^{\|M\|}$.
- *For all* $t \in \mathbb{R}$, *we have* $\frac{d}{dt} e^{tM} = Me^{tM} = e^{tM} M$.
- *Given* $x_0 \in \mathbb{R}^n$, *let* $x\colon [0,\infty) \to \mathbb{R}^n$ *be the solution to*

$$\dot{x}(t) = Mx(t), \quad x(0) = x_0.$$

Then $x(t) = e^{tM} x_0$, *for all* $t \geq 0$.

State equation

A pair $(y, u) \in W^{1,\infty}(0, T; \mathbb{R}^n) \times L^\infty(0, T; \mathbb{R}^m)$ satisfies the **state equation**: $\dot{y}(t) = Ay(t) + Bu(t)$, $y(0) = y_0$ if and only if

$$y(t) = y_0 + \int_0^t \Big(Ay(s) + Bu(s) \Big) \mathrm{d}s, \quad \forall t \in [0, T]. \tag{2}$$

Theorem 3 (Picard-Lindelöf / FR: Cauchy-Lipschitz)

Given $y_0 \in \mathbb{R}^n$ and $u \in L^\infty(0, T; \mathbb{R}^m)$, there exists a unique y satisfying (2). Moreover,

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} Bu(s) ds. \quad \textit{[Duhamel's formula]}$$

Notation: $y[u]$.

Reachable set

Some notation:

- $L^\infty(0, T; U)$: set of measurable functions from $(0, T)$ to U,
- $\bar{T}$: the value of problem (P) ($\bar{T} = \infty$ if (P) is infeasible).

Definition 4

Given $t \geq 0$, the **reachable set** at time t, $\mathcal{R}(t)$, is defined by

$$\mathcal{R}(t) = \big\{ y[u](t) \,|\, u \in L^\infty(0, t; U) \big\}.$$

Lemma 5

- *For all* $T \geq 0$, *the set* $\cup_{0 \leq t \leq T} \mathcal{R}(t)$ *is* bounded.
- *For all* $t \geq 0$, *the reachable set* $\mathcal{R}(t)$ *is* convex.

Proof. Exercise (use Duhamel's formula and boundedness of U).

Weak compactness

Definition 6

Let F be a Banach space. Let $(e_k)_{k\in\mathbb{N}}$ be a sequence in F. The sequence **converges weakly** to $\bar{e} \in F$ (notation: $e_k \rightharpoonup \bar{e}$) if

$$L(e_k) \to L(\bar{e}), \quad \text{for all continuous and linear map } L\colon F \to \mathbb{R}.$$

Remark. If $e_k \rightharpoonup \bar{e}$, then $L(e_k) \to L(\bar{e})$ for any continuous and linear map $L\colon F \to \mathbb{R}^k$.

Lemma 7

Let E be a closed and convex subset of a Hilbert space F. Let $(e_k)_{k\in\mathbb{N}}$ be a bounded sequence in E. Then there exists a weakly convergent subsequence $(e_{k_q})_{q\in\mathbb{N}}$ with weak limit in E.

Proof. See Corollary 3.22 and Proposition 5.1 in *Functional Analysis*, by H. Brézis.

Closedness of the reachable set

Lemma 8 (Closedness lemma)

Let $(\tau_k)_{k\in\mathbb{N}}$ be a convergent sequence of positive real numbers with limit $\bar{\tau} \geq 0$. Assume that $\tau_k \geq \bar{\tau}$, $\forall k \in \mathbb{N}$.
Let $(y_k)_{k\in\mathbb{N}}$ be a convergent sequence in $\mathbb{R}^n$ with limit $\bar{y}$. Assume that

$$y_k \in \mathcal{R}(\tau_k), \quad \forall k \in \mathbb{N}.$$

Then $\bar{y} \in \mathcal{R}(\bar{\tau})$.

Corollary 9

For all $t \geq 0$, the set $\mathcal{R}(t)$ is closed.

Proof of the closedness lemma

Proof. Step 1. For all $k \in \mathbb{N}$, let $u_k \in L^\infty(0, \tau_k; U)$ be such that $y[u_k](\tau_k) = y_k$. As a consequence of Lemma 5, there exists $M > 0$ (independent of k) such that

$$\|\dot{y}[u_k]\|_{L^\infty(0,\tau_k;\mathbb{R}^m)} \leq M.$$

Thus $y[u_k](\cdot)$ is M-Lipschitz, that is

$$\|y[u_k](t_2) - y[u_k](t_1)\| \leq M|t_2 - t_1|, \quad \forall t_1,\, t_2 \in [0, T].$$

Next, we have

$$\|y[u_k](\bar{\tau}) - \bar{y}\| \leq \underbrace{\|y[u_k](\bar{\tau}) - y[u_k](\tau_k)\|}_{\leq M|\tau_k - \bar{\tau}|} + \|\underbrace{y[u_k](\tau_k)}_{y_k} - \bar{y}\| \to 0.$$

Thus $y[u_k](\bar{\tau}) \to \bar{y}$.

Proof of the closedness lemma

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Thus $y[u_k](\bar{\tau}) \to \bar{y}$.

Proof of the closedness lemma

Step 2. Consider the linear map $L\colon u \in L^2(0,\bar\tau;\mathbb{R}^m) \to \mathbb{R}^n$ defined by

$$L(u) = \int_0^{\bar\tau} e^{(\bar\tau - s)A} Bu(s)\, \mathrm{d}s.$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} |L(u)| &\leq \int_0^{\bar\tau} e^{(\bar\tau - s)\|A\|} \cdot \|B\| \cdot \|u(s)\|\, \mathrm{d}s \\ &\leq \underbrace{\|B\| \cdot \left(\int_0^{\bar\tau} e^{2(\bar\tau - s))\|A\|}\, \mathrm{d}s \right)^{1/2}}_{<\infty} \|u\|_{L^2(0,\bar\tau;\mathbb{R}^m)}. \end{aligned}$$

This proves that the linear form L is **continuous**.

Proof of the closedness lemma

Step 3. Apply Lemma 7:

- $L^2(0, \bar{\tau}; \mathbb{R}^m)$ is a Hilbert space
- $L^\infty(0, \bar{\tau}; U)$ is convex, closed, and bounded.

Then the sequence u_k (restricted to $(0, \bar{\tau})$) has a weakly convergent subsequence, with limit $\bar{u}$.

We have:

$$\underbrace{y[u_{k_q}](\bar{\tau})}_{\longrightarrow \bar{y}} = e^{\bar{\tau}A} y_0 + \int_0^{\bar{\tau}} e^{(\bar{\tau}-s)A} B u_{k_q}(s) ds$$

$$= e^{\bar{\tau}A} y_0 + L(u_{k_q}) \longrightarrow e^{\bar{\tau}A} y_0 + L(\bar{u}) = y[\bar{u}](\bar{\tau}),$$

proving that $\bar{y} = y[\bar{u}](\bar{\tau}) \in \mathcal{R}(\bar{\tau})$.

Proof of the closedness lemma

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proving that $\bar{y} = y[\bar{u}](\bar{\tau}) \in \mathcal{R}(\bar{\tau})$.

Existence result

Theorem 10

Assume that $\bar{T} < \infty$. There exists an optimal control, that is, there exists $\bar{u}$ such that

$$y[\bar{u}](\bar{T}) \in C.$$

Proof. Consider the set of times at which the target can be reached, that is:

$$\mathcal{T} = \{ T \geq 0 \,|\, \mathcal{R}(T) \cap C \neq \emptyset \}.$$

By assumption $\mathcal{T}$ is non empty. By definition, $\bar{T} = inf\, \mathcal{T}$.
Our task: proving that $\bar{T} \in \mathcal{T}$.

Existence result

- It suffices to show that $\mathcal{R}(\bar{T}) \cap C \neq \emptyset$.
- Let $\tau_k \downarrow \bar{T}$ be such that for all $k \in \mathbb{N}$, there exists $y_k \in \mathcal{R}(\tau_k) \cap C$. By Lemma 5, $(y_k)_{k \in \mathbb{N}}$ is bounded. Thus it has an **accumulation point** $\bar{y}$.
- Since C is closed, $\bar{y} \in C$. By Lemma 8, $\bar{y} \in \mathcal{R}(\bar{T})$.

Methodology

For proving the optimality conditions (in the form of a Pontryagin's principle), we proceed as follows:

- Fix an optimal control $\bar{u}$ for the time-optimal problem.
- Show that $\bar{u}$ is optimal for another problem, easier to treat, referred to as auxiliary problem.
- Establish Pontryagin's principle for the auxiliary problem.

Hahn-Banach lemma

Lemma 11

Let C_1 and C_2 be two closed and convex sets of $\mathbb{R}^n$, let C_2 be bounded. Assume that $C_1 \cap C_2 = \emptyset$. Then, there exists $q \in \mathbb{R}^n \backslash \{0\}$ such that

$$\langle q, y_1 \rangle \leq \langle q, y_2 \rangle, \quad \forall y_1 \in C_1, \ \forall y_2 \in C_2.$$

We say that q separates *C_1 and C_2.*

Proof. See Brezis, Theorem 1.7.

Remark. With loss of generality, we can assume that $\|q\| = 1$.

Hahn-Banach lemma

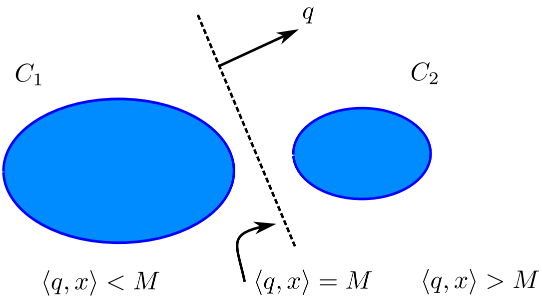

Figure: Illustration of Hahn-Banach lemma.

Normal cones

Definition 12

Let K be a subset of $\mathbb{R}^n$ and let $x \in K$. The normal cone of K at x, denoted $N_K(x)$ is defined by

$$N_K(x) = \{q \in \mathbb{R}^n \mid \langle q, y - x \rangle \leq 0, \ \forall y \in K\}.$$

Some examples.

- If $K = \{\bar{x}\}$, then $N_K(\bar{x}) = \mathbb{R}^n$.
- If $K = \mathbb{R}^n$, then $N_K(x) = \{0\}$ for any $x \in \mathbb{R}^n$.
- Let $\mathbb{R}^n_{\geq 0} := \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, ..., n\}$.
  Let $\mathbb{R}^n_{\leq 0} := \{x \in \mathbb{R}^n \mid x_i \leq 0, i = 1, ..., n\}$. Then

$$N_{\mathbb{R}^n_{\geq 0}}(0) = \mathbb{R}^n_{\leq 0} \quad \text{and} \quad N_{\mathbb{R}^n_{\leq 0}}(0) = \mathbb{R}^n_{\geq 0}$$

Normal cones

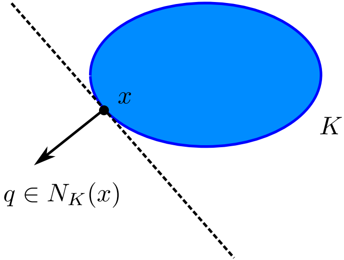

Figure: A vector in the normal cone.

A separation result

Lemma 13 (Separation lemma)

Let $\bar{T}$ denote the value of the time optimal control problem (P). Assume that $0 < \bar{T} < \infty$. Then, there exists $\bar{q} \in \mathbb{R}^n \setminus \{0\}$ such that

$$\langle \bar{q}, z \rangle \leq \langle \bar{q}, y \rangle, \quad \forall z \in C, \quad \forall y \in \mathcal{R}(\bar{T}).$$

Corollary 14

For any optimal control $\bar{u}$, *we have* $\bar{q} \in N_C(y[\bar{u}](\bar{T}))$.

Proof of the corollary. Take $y = y[\bar{u}](\bar{T})$ in the separation lemma.

A separation result

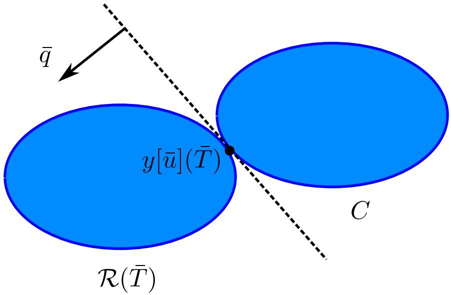

Figure: Illustration of the separation lemma.

A separation result

Proof of the separation lemma.

- Let $T_k \uparrow \bar{T}$. For all $k \in \mathbb{N}$, $\mathcal{R}(T_k) \cap C = \emptyset$.
- The set C is convex and closed, $\mathcal{R}(T_k)$ is compact and convex (by Lemma 5 and Lemma 8).
- By the Hahn-Banach Lemma, there exists q_k such that $\|q_k\| = 1$ and

$$\langle q_k, z\rangle \leq \langle q_k, y\rangle, \quad \forall z \in C,\ \forall y \in \mathcal{R}(T_k). \tag{3}$$

Extracting a subsequence if necessary, we assume that $q_k \to \bar{q}$ for some $\bar{q} \in \mathbb{R}^n$ with $\|\bar{q}\| = 1$.

A separation result

We next show that $\bar{q}$ separates C and $\mathcal{R}(\bar{T})$.

- Let $z \in C$ and let $y \in \mathcal{R}(\bar{T})$.
  Let $u \in L^\infty(0, T; U)$ be such that $y[u](\bar{T}) = y$.
  Set $y_k = y[u](T_k) \in \mathcal{R}(T_k)$.
- Inequality (3) yields:

$$\langle q_k, z \rangle \leq \langle q_k, y_k \rangle, \quad \forall k \in \mathbb{N}.$$

- We pass to the limit and obtain

$$\langle \bar{q}, z \rangle \leq \langle \bar{q}, y \rangle.$$

An auxiliary problem

Let $T > 0$, let $y_0 \in \mathbb{R}^n$, and let $q \in \mathbb{R}^n$ be fixed.
Consider the following **auxiliary** optimal control problem:

$$\inf_{\substack{y \in W^{1,\infty}(0,T;\mathbb{R}^n) \\ u \in L^\infty(0,T;U)}} \langle q, y(T) \rangle, \quad \text{s.t.:} \begin{cases} \dot{y}(t) = & Ay(t) + Bu(t) \\ y(0) = & y_0, \\ u(t) \in & U. \end{cases} \qquad (P_{\text{aux}}[q,T])$$

Remark: Let $(\bar{u}, \bar{y}, \bar{T})$ be a solution to the time-optimal problem. Let $\bar{q}$ be as in the separation lemma. Then $(\bar{u}, \bar{y})$ is a solution to $P_{\text{aux}}[q,T]$, with $(q,T) = (\bar{q}, \bar{T})$.

Pre-Hamiltonian and adjoint equation

Define the **pre-Hamiltonian**:

$$H\colon (u, y, p) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \langle p, Ay + Bu \rangle \in \mathbb{R}.$$

Note that

$$H(u, y, p) = \langle A^\top p, y \rangle + \langle B^\top p, u \rangle.$$

Thus,

$$\nabla_y H(u, y, p) = A^\top p \quad \text{and} \quad \nabla_u H(u, y, p) = B^\top p.$$

Let us define p as the solution to the **adjoint equation** (also called costate equation):

$$\begin{cases} p(T) = q \\ -\dot{p}(t) = A^\top p(t) = \nabla_y H(p(t)). \end{cases} \tag{4}$$

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Pontryagin's principle

Theorem 15 (Pontryagin's minimum principle)

Let $(\bar{y}, \bar{u})$ be such that $\bar{y} = y[\bar{u}]$.
Then $(\bar{y}, \bar{u})$ is a solution *to $(P_{\text{aux}}[q, T])$ if and only if*

$$\bar{u}(t) \in \operatorname*{argmin}_{v \in U} H(v, \bar{y}(t), p(t)), \quad \text{for a.e. } t \in (0, T).$$

Remark:

$$\operatorname*{argmin}_{v \in U} H(v, \bar{y}(t), p(t)) = \operatorname*{argmin}_{v \in U} \langle B^\top p(t), v \rangle.$$

Proof of Pontryagin's principle

"$\Longleftarrow$" Assume that $(\bar{y}, \bar{u})$ satisfies Pontryagin's principle.
Let (y, u) be such that $y = y[u]$. Then

$$\langle q, y(T) - \bar{y}(T) \rangle = \langle p(T), y(T) - \bar{y}(T) \rangle - \langle p(0), \underbrace{y(0) - \bar{y}(0)}_{= y_0 - y_0 = 0} \rangle$$

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$$= \int_0^T \frac{d}{dt}\langle p(t), y(t) - \bar{y}(t)\rangle \, \mathrm{d}t$$

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$$= \int_0^T \frac{d}{dt}\langle p(t), y(t) - \bar{y}(t)\rangle \, \mathrm{d}t$$

$$= \int_0^T \langle \dot{p}(t), y(t) - \bar{y}(t)\rangle + \int_0^T \langle p(t), \dot{y}(t) - \dot{\bar{y}}(t)\rangle \, \mathrm{d}t$$

Proof of Pontryagin's principle

"$\Longleftarrow$" Assume that $(\bar{y}, \bar{u})$ satisfies Pontryagin's principle.
Let (y, u) be such that $y = y[u]$. Then

$$
\begin{aligned}
\langle q, y(T) - \bar{y}(T)\rangle &= \langle p(T), y(T) - \bar{y}(T)\rangle - \langle p(0), \underbrace{y(0) - \bar{y}(0)}_{=y_0 - y_0 = 0}\rangle \\
&= \int_0^T \frac{d}{dt}\langle p(t), y(t) - \bar{y}(t)\rangle \, \mathrm{d}t \\
&= \int_0^T \langle \dot{p}(t), y(t) - \bar{y}(t)\rangle + \int_0^T \langle p(t), \dot{y}(t) - \dot{\bar{y}}(t)\rangle \, \mathrm{d}t \\
&= \int_0^T \langle -A^\top p(t), y(t) - \bar{y}(t)\rangle \, \mathrm{d}t \\
&\quad + \int_0^T \langle p(t), Ay(t) + Bu(t) - A\bar{y}(t) - B\bar{u}(t)\rangle \, \mathrm{d}t
\end{aligned}
$$

Proof of Pontryagin's principle

"$\Longleftarrow$" Assume that $(\bar{y}, \bar{u})$ satisfies Pontryagin's principle.
Let (y, u) be such that $y = y[u]$. Then

$$\langle q, y(T) - \bar{y}(T)\rangle = \langle p(T), y(T) - \bar{y}(T)\rangle - \langle p(0), \underbrace{y(0) - \bar{y}(0)}_{=y_0 - y_0 = 0}\rangle$$

$$= \int_0^T \frac{d}{dt}\langle p(t), y(t) - \bar{y}(t)\rangle \, \mathrm{d}t$$

$$= \int_0^T \langle \dot{p}(t), y(t) - \bar{y}(t)\rangle + \int_0^T \langle p(t), \dot{y}(t) - \dot{\bar{y}}(t)\rangle \, \mathrm{d}t$$

$$= \int_0^T \langle -p(t), Ay(t) - A\bar{y}(t)\rangle \, \mathrm{d}t$$

$$+ \int_0^T \langle p(t), Ay(t) + Bu(t) - A\bar{y}(t) - B\bar{u}(t)\rangle \, \mathrm{d}t$$

Proof of Pontryagin's principle

"$\Longleftarrow$" Assume that $(\bar{y}, \bar{u})$ satisfies Pontryagin's principle.
Let (y, u) be such that $y = y[u]$. Then

$$\begin{aligned}
\langle q, y(T) - \bar{y}(T)\rangle &= \langle p(T), y(T) - \bar{y}(T)\rangle - \langle p(0), \underbrace{y(0) - \bar{y}(0)}_{=y_0 - y_0 = 0}\rangle \\
&= \int_0^T \frac{d}{dt}\langle p(t), y(t) - \bar{y}(t)\rangle \, \mathrm{d}t \\
&= \int_0^T \langle \dot{p}(t), y(t) - \bar{y}(t)\rangle + \int_0^T \langle p(t), \dot{y}(t) - \dot{\bar{y}}(t)\rangle \, \mathrm{d}t \\
&= \int_0^T \langle -p(t), Ay(t) - A\bar{y}(t)\rangle \, \mathrm{d}t \\
&\quad + \int_0^T \langle p(t), Ay(t) + Bu(t) - A\bar{y}(t) - B\bar{u}(t)\rangle \, \mathrm{d}t \\
&= \int_0^T \langle B^\top p(t), u(t) - \bar{u}(t)\rangle \, \mathrm{d}t \geq 0.
\end{aligned}$$

Proof of Pontryagin's principle

"$\Longrightarrow$" Assume that $(\bar{y}, \bar{u})$ is optimal. Consider the time function

$$h \colon t \in [0, T] \mapsto \langle B^\top p(t), \bar{u}(t) \rangle \in \mathbb{R}.$$

A time t is called Lebesgue point if

$$h(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} h(s) \mathrm{d}s.$$

Lebesgue differentiation theorem states that almost every time t is a Lebesgue function, since $h \in L^1(0, T)$.

Let t be a Lebesgue point. Let $v \in U$. Let u_ε be defined by

$$u_\varepsilon(s) = \begin{cases} v & \text{if } s \in (t - \varepsilon, t + \varepsilon) \\ \bar{u}(s) & \text{otherwise.} \end{cases}$$

Proof of Pontryagin's principle

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Let t be a Lebesgue point. Let $v \in U$. Let u_ε be defined by

$$u_\varepsilon(s) = \begin{cases} v & \text{if } s \in (t-\varepsilon, t+\varepsilon) \\ \bar{u}(s) & \text{otherwise.} \end{cases}$$

Proof of Pontryagin's principle

The same calculation as above leads to:

$$\begin{aligned} 0 &\leq \frac{1}{2\varepsilon}\langle q, y[u_\varepsilon](T) - \bar{y}(T)\rangle \\ &= \frac{1}{2\varepsilon}\int_0^T \langle B^\top p(s), u_\varepsilon(t) - \bar{u}(s)\rangle \, \mathrm{d}s \\ &= \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \langle B^\top p(s), v - \bar{u}(s) \, \mathrm{d}s \\ &\underset{\varepsilon \downarrow 0}{\longrightarrow} \langle B^\top p(t), v - \bar{u}(t)\rangle, \end{aligned}$$

as was to be proved.

Pontryagin for time-optimal problems

We come back to the **time-optimal control** problem (P).

Theorem 16 (Pontryagin's principle)

Let $y_0 \notin C$, assume that $\bar{T} < \infty$. Let $(\bar{y}, \bar{u})$ be a solution to the original minimum time problem (P).
Then, there exists $\bar{q} \in N_C(\bar{y}(\bar{T}))$, $\bar{q} \neq 0$ such that

$$\bar{u}(t) \in \underset{v \in U}{\text{argmin}}\ H(v, \bar{y}(t), p(t)) = \underset{v \in U}{\text{argmin}}\ \langle B^\top p, v \rangle, \tag{5}$$

where p is the solution to the costate equation:

$$-\dot{p}(t) = A^\top p(t), \quad p(\bar{T}) = \bar{q}.$$

Remark. Pontryagin's principle is only a necessary optimality condition.

Proof

Proof.

- By Lemma 13 and by Corollary 14, there exists $\bar{q} \in N_C(\bar{y}(\bar{T}))$ such that

$$\langle \bar{q}, z \rangle \leq \langle \bar{q}, y \rangle, \quad \forall z \in C, \quad \forall y \in \mathcal{R}(\bar{T}).$$

  We take $z = \bar{y}(\bar{T}) \in C$.

- It follows that $(\bar{y}, \bar{u})$ is a **solution to the auxiliary problem** $(P_{\text{aux}}[q, T])$, with $q = \bar{q}$ and $T = \bar{T}$.
- Applying Pontryagin's principle to the auxiliary problem (Theorem 15), we obtain (5).

Lunar landing problem

Recall the problem:

$$\inf_{\substack{T\geq 0\\ h:[0,T]\to\mathbb{R}\\ v:[0,T]\to\mathbb{R}\\ u:[0,T]\to\mathbb{R}}} T, \quad \text{s.t.:} \begin{cases} \dot{h}(t) = v(t), & h(0) = h_0, \quad h(T) = 0 \\ \dot{v}(t) = u(t), & v(0) = v_0, \quad v(T) = 0 \\ u(t) \in [-1,1]. \end{cases}$$

The dynamics writes:

$$\begin{pmatrix} \dot{h}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t).$$

The lunar landing problem is a special case of (P), with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \{0\}.$$

Lunar landing problem

We apply Pontryagin's principle. Let T be the optimal time.

- Costate equation (4) reads:

$$-\begin{pmatrix} \dot{p}_h(t) \\ \dot{p}_v(t) \end{pmatrix} = A^\top \begin{pmatrix} p_h(t) \\ p_v(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_h(t) \\ p_v(t) \end{pmatrix} = \begin{pmatrix} 0 \\ p_h(t) \end{pmatrix}.$$

- Terminal condition: $(p_h(T), p_v(T)) \in N_C(\bar{h}(T), \bar{v}(T)) = \mathbb{R}^2$ does not bring any information!

- Analytic resolution:

$$p_h(t) = p_h(T), \quad \dot{p}_v(t) = -p_h(t) = -p_h(T)$$

and thus

$$p_v(t) = p_v(T) + p_h(T)(T - t).$$

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- Terminal condition: $(p_h(T), p_v(T)) \in N_C(\bar{h}(T), \bar{v}(T)) = \mathbb{R}^2$ does not bring any information!
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Lunar landing problem

- The minimization condition reads:

$$\bar{u}(t) \in \underset{v \in [-1,1]}{\operatorname{argmin}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^\top \begin{pmatrix} p_h(t) \\ p_v(t) \end{pmatrix} v = \underset{v \in [-1,1]}{\operatorname{argmin}} \; p_v(t) v.$$

It follows that

$$\begin{cases} \bar{u}(t) = -1 & \text{if } p_v(t) > 0 \\ \bar{u}(t) = 1 & \text{if } p_v(t) < 0 \end{cases} \quad \text{for a.e. } t \in [0, T].$$

Lunar landing problem

We now **prove the original conjecture**: any optimal control is piecewise constant, with at most two pieces, taking values in $\{-1, 1\}$.

- Case 1: $p_h(T) = 0$. Then $p_v(T) \neq 0$. Therefore
  - either $p_v(t) = p_v(T) < 0 \Longrightarrow \bar{u}(t) = 1$
  - or $p_v(t) = p_v(T) > 0 \Longrightarrow \bar{u}(t) = -1$.
- Case 2: $p_h(T) \neq 0$. Then the map $t \mapsto p_v(T) + p_h(T)(T - t)$ vanishes at exactly one point, say τ.
  - If $\tau \leq 0$ or $\tau \geq T$, then the optimal control is constant, equal to 1 or -1.
  - If $\tau \in (0, T)$, then there is a switch.

Lunar landing problem

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  - If $\tau \leq 0$ or $\tau \geq T$, then the optimal control is constant, equal to 1 or -1.
  - If $\tau \in (0, T)$, then there is a switch.

A summary

Given a linear time-optimal control problem, the following methodology can be followed to analyze it:

1. Put the state equation in the form $\dot{y} = Ay + Bu$. Check the **assumptions** state at the beginning of Section 2.
2. **Existence** of a solution: verify the applicability of Theorem 10.
3. Derive **optimality conditions** with Theorem 15.
4. Deduce **structural properties** of optimal controls and trajectories.
5. Transform the problem into a **geometric** problem.
6. Solve it!