Optimal Control of Ordinary Differential Equations
SOD 311

Laurent Pfeiffer
Inria and Ecole Polytechnique, Institut Polytechnique de Paris

Ensta-Paris
Paris-Saclay University
October 21, 2021
Lecture 3:
HJB equation and viscosity solutions

- **Goal**: finding global solutions to optimal control problems (in feedback form), by solving a non-linear PDE.
- **Issues**: characterization of the value function with the Hamilton-Jacobi-Bellman equation.
The following references are related to Chapter 3:


1. Introduction

2. Dynamic programming principle

3. A first characterization of the value function

4. HJB equation: the classical sense

5. HJB equation: viscosity solutions
Our results so far were based on optimality conditions (Pontryagin’s principle).

Now: a different approach, based on dynamic programming. In some sense, more specific to optimal control.

The dynamic programming principle is ubiquitous in optimization. A very general concept allowing to “split” some problems into a family of simpler problems.

The central tool: the value function \( V \).

- Defined as the value of the optimization problem, expressed as a function of the initial state.
- Characterized as the unique viscosity solution of a non-linear partial differential equation (PDE) called HJB equation.
Introduction

- **Interest**: a **globally optimal** solution to the problem can be derived from $V$.

- **Limitation**: **curse of dimensionality**.

- **Warning**: focus on a specific class of problems. All concepts can be extended, in particular to a stochastic framework (finance), and to other nonlinear PDEs.
Problem formulation

**Data of the problem and assumptions:**

- A parameter $\lambda > 0$.
- A non-empty and compact subset $U$ of $\mathbb{R}^m$.
- A mapping $f : (u, y) \in U \times \mathbb{R}^n \to \mathbb{R}^n$, such that
  \[
  \|f(u, y)\| \leq \|f\|_\infty, \\
  \|f(u', y') - f(u, y)\| \leq L_f\|(u', y') - (u, y)\|,
  \]
  for all $(u, y)$ and $(u', y') \in U \times \mathbb{R}^n$.
- A mapping $\ell : (u, y) \in U \times \mathbb{R}^n \to \mathbb{R}$, such that
  \[
  \|\ell(u, y)\| \leq \|\ell\|_\infty, \\
  \|\ell(u', y') - \ell(u, y)\| \leq L_\ell\|(u', y') - (u, y)\|,
  \]
  for all $(u, y)$ and $(u', y') \in U \times \mathbb{R}^n$. 
**Problem formulation**

- **Notation:** for any $\tau \in [0, \infty]$, $U_\tau$ is the set of measurable functions from $(0, \tau)$ to $U$.

- State equation: for $x \in \mathbb{R}^n$ and $u \in U_\infty$, there is a unique solution $y[u, x]$ to the ODE

$$
\dot{y}(t) = f(u(t), y(t)), \quad y(0) = x,
$$

by the theorem of Picard-Lindelöf (Cauchy-Lipschitz).

- Cost function $W$, for $u \in U_\infty$ and $x \in \mathbb{R}^n$:

$$
W(u, x) = \int_0^\infty e^{-\lambda t} \ell(u(t), y[u, x](t)) \, dt.
$$

- Optimal control problem and value function $V$:

$$
V(x) = \inf_{u \in U_\infty} W(u, x). \quad (P(x))
$$
Lemma 1 (Grönwall’s lemma)

Let $\alpha > 0$ and let $\beta > 0$. Let $\theta : [0, \infty) \to \mathbb{R}$ be a continuous function such that

$$\theta(t) \leq \alpha + \beta \int_0^t \theta(s) \, ds, \quad \forall t \in [0, \infty).$$

Then, $\theta(t) \leq \alpha e^{\beta t}$, for all $t \in [0, \infty)$.

Corollary 2

Let $u \in \mathcal{U}_\infty$. For all $x$ and $\tilde{x}$, for all $t \geq 0$, it holds:

$$\|y[u, x](t) - y[u, \tilde{x}](t)\| \leq e^{L_f t} \|x - \tilde{x}\|.$$

Proof. Grönwall with $\theta = \|y - \tilde{y}\|$, $\alpha = \|x - \tilde{x}\|$, $\beta = L_f$. 
1 Introduction

2 Dynamic programming principle

3 A first characterization of the value function

4 HJB equation: the classical sense

5 HJB equation: viscosity solutions
Introduction

Dynamic programming

1st characterization

Classical HJB

Viscosity solutions

Dynamic programming principle

Theorem 3 (Dynamic programming (DP) principle)

Let $\tau > 0$. Then for all $x \in \mathbb{R}^n$, abbreviating $y = y[u, x]$,

$$V(x) = \inf_{u \in U_\tau} \left( \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, dt + e^{-\lambda \tau} V(y(\tau)) \right). \quad (DPP)$$

Interpretation:

- $V(x)$ is the value function of an optimal control problem on the interval $(0, \tau)$.
- The original integral has been truncated:

$$\int_\tau^\infty e^{-\lambda t} \ell(u(t), y(t)) \, dt \sim e^{-\lambda \tau} V(y(\tau)).$$

The term $e^{-\lambda \tau} V(y(\tau))$ is the “optimal cost from $\tau$ to $\infty$”. 
Lemma 4 (Flow property)

Let $x \in \mathbb{R}^n$ and let $u \in \mathcal{U}_\infty$. Define:

- $u_1 = u|_{(0,\tau)} \in \mathcal{U}_\tau$
- $u_2 = u|_{(\tau,\infty)} \in L^\infty(\tau, \infty; U)$
- $\tilde{u}_2 \in \mathcal{U}_\infty$, $\tilde{u}_2(t) = u_2(t + \tau)$.

It holds:

$$y[u, x](t) = y[\tilde{u}_2, y[u_1, x](\tau)](t - \tau),$$

for any $t \geq \tau$.

Remark. After time $\tau$, one can forget $u_1$ and only remember $y[x, u_1](\tau)$. 
Proof of the DP-principle. Let us denote

\[ \tilde{V}(x) = \inf_{u \in U_{\tau}} \left( \int_{0}^{\tau} e^{-\lambda t} \ell(u(t), y(t)) \, dt + e^{-\lambda \tau} V(y(\tau)) \right). \]

Step 1: \( V \geq \tilde{V} \). Let \( u, u_1, u_2, \) and \( \tilde{u}_2 \) be as in Lemma 4.

\[ W(u, x) = \int_{0}^{\infty} e^{-\lambda t} \ell(u(t), y[u, x](t)) \, dt \]

\[ = \int_{0}^{\tau} e^{-\lambda t} \ell(u(t), y[u, x](t)) \, dt \]

\[ + e^{-\lambda \tau} \int_{\tau}^{\infty} e^{-\lambda (t-\tau)} \ell(u(t), y[u, x](t)) \, dt \]

\[ = \int_{0}^{\tau} e^{-\lambda t} \ell(u(t), y[u, x](t)) \, dt \]

\[ + e^{-\lambda \tau} \int_{0}^{\infty} e^{-\lambda s} \ell(u(s + \tau), y[u, x](s + \tau)) \, ds. \]
We further have, for the last integral:

\[
\int_0^\infty e^{-\lambda s} \ell(u(s + \tau), y[u, x](s + \tau)) \, ds
\]

\[
= \int_0^\infty e^{-\lambda s} \ell(\tilde{u}_2(s), y[\tilde{u}_2, y[u_1, x](\tau)](s)) \, ds
\]

\[
= \mathcal{W}(\tilde{u}_2, y[u_1, x](\tau)) \geq V(y[u_1, x](\tau)).
\]

Injecting in the above equality:

\[
\mathcal{W}(u, x) \geq \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1, x](t)) \, dt + e^{-\lambda \tau} V(y[u_1, x](\tau))
\]

\[
\geq \tilde{V}(x).
\]

Minimizing with respect to \( u \) yields \( V \geq \tilde{V} \).
Step 2: $\tilde{V} \leq V$. Let $\varepsilon > 0$. Let $u_1 \in \mathcal{U}_\tau$ be such that

$$\int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1](t)) \, dt + e^{-\lambda \tau} V(y[u_1, x](\tau)) \leq \tilde{V}(x) + \varepsilon / 2.$$ 

Let $\tilde{u}_2 \in \mathcal{U}_\infty$ be such that

$$W(\tilde{u}_2, y[u_1, x](\tau)) \leq V(y[u_1, x](\tau)) + \varepsilon / 2.$$ 

Let $u$ be defined by

$$u(t) = \begin{cases} 
    u_1(t) & \text{for a.e. } t \in (0, \tau), \\
    \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty). 
\end{cases}$$
Proof

The same calculation as above yields:

\[ W(u,x) = \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1,x](t)) \, dt \]

\[ + e^{-\lambda \tau} \int_0^\infty e^{-\lambda t} \ell(\tilde{u}_2(t), y[\tilde{u}_2(t), y[u_1,x](\tau)](t)) \, dt \]

\[ = W(\tilde{u}_2,y[u_1,x](\tau)) \]

\[ \leq \int_0^\tau e^{-\lambda t} \ell(u_1(t), y[u_1,x](t)) \, dt \]

\[ + e^{-\lambda \tau} (V(y[u_1,x](\tau)) + \varepsilon/2) \]

\[ \leq \tilde{V}(x) + \varepsilon. \]

It follows that

\[ V(x) \leq \tilde{V}(x) + \varepsilon, \quad \forall \varepsilon > 0. \]
Corollary 5

Let \( u \in \mathcal{U}_\infty \) be a solution to \( P(x) \). Let \( \tau > 0 \). Let \( u_1 \) and \( \tilde{u}_2 \) be defined as in Lemma 4. Then,

- \( u_1 \) is optimal in the DP principle
- \( \tilde{u}_2 \) is optimal for \( P(y[u_1, x](\tau)) \).

Conversely: let \( u_1 \) be a minimizer of \((DPP)\). Let \( \tilde{u}_2 \) be a solution to \( P(y[u_1, x])(\tau) \). Let \( u \in \mathcal{U}_\infty \) be defined by

\[
  u(t) = \begin{cases} 
    u_1(t) & \text{for a.e. } t \in (0, \tau) \\
    \tilde{u}_2(t - \tau) & \text{for a.e. } t \in (\tau, \infty).
  \end{cases}
\]

Then \( u \) is a solution to \( P(x) \).

What can we do with the value function? If \( \mathcal{V} \) is known, then the DP-principle allows to **decouple** the problem in time.
1 Introduction

2 Dynamic programming principle

3 A first characterization of the value function

4 HJB equation: the classical sense

5 HJB equation: viscosity solutions
The value function $V$ is bounded. It is also uniformly continuous, that is, for all $\varepsilon > 0$, there exists $\alpha > 0$ such that for all $x$ and $\tilde{x} \in \mathbb{R}^n$,

$$\|\tilde{x} - x\| \leq \alpha \implies |V(\tilde{x}) - V(x)| \leq \varepsilon.$$  

Proof. Step 1: proof of boundedness. Let $x \in \mathbb{R}^n$ and $u \in U_\infty$. We have

$$|W(x, u)| \leq \int_0^\infty e^{-\lambda t} \|\ell\|_\infty \, dt \leq \frac{1}{\lambda} \|\ell\|_\infty.$$  

Thus $|V(x)| \leq \frac{1}{\lambda} \|\ell\|_\infty$. 


Step 2: proof of uniform continuity. Let $\varepsilon > 0$. Let $\alpha > 0$. Let $x$ and $\tilde{x}$ be such that $\|\tilde{x} - x\| \leq \alpha$, we will specify $\alpha$ later. We have:

$$|V(\tilde{x}) - V(x)| = \left| \inf_{u \in U_\infty} W(\tilde{x}, u) - \inf_{u \in U_\infty} W(x, u) \right|$$

$$\leq \sup_{u \in U_\infty} \left| W(\tilde{x}, u) - W(x, u) \right| \leq \Delta_1 + \Delta_2,$$

where

$$\Delta_1 = \sup_{u \in U_\infty} \int_{0}^{\tau} e^{-\lambda t} \left| \ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t)) \right| dt$$

$$\Delta_2 = \sup_{u \in U_\infty} \int_{\tau}^{\infty} e^{-\lambda t} \left| \ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t)) \right| dt,$$

where $\tilde{y} = y[\tilde{x}, u]$ and $y = y[x, u]$ and where $\tau > 0$ is arbitrary.
Regularity of $V$

- **Bound of $\Delta_1$.** By Corollary 2,
  \[
  \|\tilde{y}(t) - y(t)\| \leq e^{L_f t}\|\tilde{x} - x\| \leq e^{L_f \tau} \alpha, \quad \forall t \in [0, \tau].
  \]
  Therefore, $\Delta_1 \leq \tau L_\ell e^{L_f \tau} \alpha$.

- **Bound of $\Delta_2$.** Since $\ell$ is bounded,
  \[
  \Delta_2 \leq 2\|\ell\|_\infty \int_\tau^\infty e^{-\lambda t} \, dt = 2\|\ell\|_\infty e^{-\lambda \tau}.
  \]

**Conclusion:** take $\tau > 0$ sufficiently large, so that $\Delta_2 \leq \frac{\varepsilon}{2}$. Take then $\alpha$ sufficiently small, so that $\Delta_1 \leq \frac{\varepsilon}{2}$. The construction of $\alpha$ is independent of $x$ and $\tilde{x}$. We have $|V(x) - V(\tilde{x})| \leq \varepsilon$. 
(More) regularity of $V$

**Lemma 7**

We have

- if $\lambda < L_f$, then $V$ is $(\lambda/L_f)$-Hölder continuous
- if $\lambda = L_f$, then $V$ is $\alpha$-Hölder continuous for all $\alpha \in (0, 1)$
- if $\lambda > L_f$, then $V$ is Lipschitz continuous.

**Exercise**: prove the last statement of the lemma.
(More) regularity of $V$

**Proof of the last case.** We have

$$|V(\tilde{x}) - V(x)| \leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} |\ell(u(t), \tilde{y}(t)) - \ell(u(t), y(t))| \, dt$$

$$\leq \sup_{u \in \mathcal{U}_\infty} \int_0^\infty e^{-\lambda t} L_\ell \|\tilde{y}(t) - y(t)\| \, dt$$

$$\leq \int_0^\infty e^{-\lambda t} L_\ell e^{L_f t} \|\tilde{x} - x\| \, dt$$

$$\leq \frac{L_\ell}{\lambda - L_f} \|\tilde{x} - x\|.$$
Notation: \( BUC(\mathbb{R}^n) \) is the set of bounded and uniformly continuous functions from \( \mathbb{R}^n \) to \( \mathbb{R} \).

**Lemma 8**

The space \( BUC(\mathbb{R}^n) \), equipped with the uniform norm (denoted \( \| \cdot \|_\infty \) ) is a Banach space.

Fix \( \tau > 0 \). Consider the “\text{DP-mapping}”: 

\[
T : v \in BUC(\mathbb{R}^n) \mapsto T v \in BUC(\mathbb{R}^n),
\]

defined by 

\[
T v(x) = \inf_{u \in \mathcal{U}_\tau} \left( \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, dt + e^{-\lambda \tau} v(y(\tau)) \right),
\]

where \( y = y[u, x] \). **Exercise:** verify that \( T v \in BUC(\mathbb{R}^n) \).
Proof. Let \( v \in BUC(\mathbb{R}^n) \). Let \( \varepsilon > 0 \). Let \( \alpha_0 > 0 \) be such that

\[
\|\tilde{x} - x\| \leq \alpha_0 \implies |v(\tilde{x}) - v(x)| \leq \varepsilon/2.
\]

Let \( \alpha > 0 \). Let \( x \) and \( \tilde{x} \in \mathbb{R}^n \) be such that \( \|\tilde{x} - x\| \leq \alpha \). The value of \( \alpha \) will be fixed later.

For all \( u \in U_\tau \), for all \( t \in [0, \tau] \), we have

\[
\|y[u, \tilde{x}](t) - y[u, x](t)\| \leq e^{L_f t}\|\tilde{x} - x\| \leq e^{L_f \tau} \alpha.
\]

We have \( \mathcal{T} v(\tilde{x}) - \mathcal{T} v(x) \leq \Delta_1 + \Delta_2 \), with...
Dynamic programming

\[ \Delta_1 = \sup_{u \in U_T} \left| \int_0^T e^{-\lambda t} \ell(u(t), y(t)) \, dt - \int_0^T e^{-\lambda t} \ell(u(t), \tilde{y}(t)) \, dt \right|, \]

\[ \Delta_2 = \sup_{u \in U_T} \left| e^{-\lambda \tau} v(\tilde{y}(\tau)) - e^{-\lambda \tau} v(y(\tau)) \right|. \]

We fix now

\[ \alpha = e^{-L_f \tau} \left( \alpha_0, \frac{\varepsilon}{2\tau} \right). \]

We have

\[ \Delta_1 \leq L_\ell e^{T L_f} \alpha \leq \varepsilon / 2 \quad \text{and} \quad \Delta_2 \leq \varepsilon_2, \]

since \( \|\tilde{y}(\tau) - y(\tau)\| \leq e^{L_f \tau} \alpha \leq \alpha_0 \). Therefore,

\[ |T v(\tilde{x}) - T v(x)| \leq \varepsilon. \]
Lemma 9

The operator $\mathcal{T}$ is Lipschitz continuous with modulus $e^{-\lambda \tau}$.

Proof. Let $x \in \mathbb{R}^n$. We have

$$|\mathcal{T}\tilde{v}(x) - \mathcal{T}v(x)| \leq \sup_{u \in \mathcal{U}_\tau} |e^{-\lambda \tau}\tilde{v}(y[x, u](\tau)) - e^{-\lambda \tau}v(y[x, u](\tau))|$$

$$\leq e^{-\lambda \tau}\|\tilde{v} - v\|_\infty.$$

We conclude that

$$\|\mathcal{T}\tilde{v} - \mathcal{T}v\|_\infty \leq e^{-\lambda \tau}\|\tilde{v} - v\|_\infty.$$
A characterization of $V$

Lemma 10

The value function $V$ is the unique solution of the fixed-point equation:

$$TV = v, \quad v \in BUC(\mathbb{R}^n).$$

Proof.

- Existence: direct consequence of the DP principle ($V = TV$).
- Uniqueness: for any $v$ such that $v = TV$, we have

$$\|v - V\|_\infty = \|TV - TV\|_\infty \leq e^{-\lambda \tau} \|v - V\|_\infty.$$  

Thus $v = V$.

Remark: the dynamic programming principle entirely characterises the value function!
**Min-plus linearity**

*Notation.* Given $v_1$ and $v_2 \in BUC(\mathbb{R}^n)$, we write $v_1 \leq v_2$ if $v_1(x) \leq v_2(x)$ for all $x \in \mathbb{R}^n$. We define $\min(v_1, v_2) \in BUC(\mathbb{R}^n)$ by

$$\min(v_1, v_2)(x) = \min(v_1(x), v_2(x)), \quad \forall x \in \mathbb{R}^n.$$ 

Given $\alpha \in \mathbb{R}$, we define $v_1 + \alpha$ by $(v_1 + \alpha)(x) = v_1(x) + \alpha$.

**Lemma 11**

*Let $v_1$ and $v_2 \in BUC(\mathbb{R}^n)$. Let $\alpha \in \mathbb{R}$. The map $\mathcal{T}$ is monotone:* 

$$v_1 \leq v_2 \implies \mathcal{T}v_1 \leq \mathcal{T}v_2$$

*and min-plus linear:*

$$\min(\mathcal{T}v_1, \mathcal{T}v_2) = \mathcal{T}\min(v_1, v_2), \quad \mathcal{T}(v + \alpha) = (\mathcal{T}v) + e^{-\lambda\tau}\alpha.$$ 

*Proof:* exercise.
1. Introduction

2. Dynamic programming principle

3. A first characterization of the value function

4. HJB equation: the classical sense

5. HJB equation: viscosity solutions
Hamiltonian

We define the **pre-Hamiltonian** $H$ and the **Hamiltonian** $\mathcal{H}$ by

$$H(u, x, p) = \ell(u, x) + \langle p, f(u, x) \rangle,$$

$$\mathcal{H}(x, p) = \min_{u \in U} H(u, x, p).$$

**Lemma 12**

*The mapping $\mathcal{H}$ is continuous, concave with respect to $p$, and Lipschitz continuous with respect to $p$ with modulus $\|f\|_{\infty}$.***

*Proof.* The pre-Hamiltonian $H$ is affine in $p$, thus concave in $p$. As an infimum of concave functions, $\mathcal{H}$ is concave. We have:

$$|\mathcal{H}(x, \tilde{p}) - \mathcal{H}(x, p)| \leq \sup_{u \in U} |H(u, x, \tilde{p}) - H(u, x, p)|$$

$$\leq \sup_{u \in U} |\langle \tilde{p} - p, f(u, x) \rangle| \leq \|\tilde{p} - p\| \cdot \|f\|_{\infty}.$$
Informal derivation

**Notation:** $C^1(\mathbb{R}^n)$, the set of continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$.

**Lemma 13**

Let $\Phi \in C^1(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$, let $u \in \mathcal{U}_\infty$, let $y = y[u, x]$. Consider the mapping:

$$
\varphi: \tau \in [0, \infty) \mapsto \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x).
$$

Then $\varphi(0) = 0$ and $\varphi \in W^{1, \infty}(0, \infty)$ with

$$
\dot{\varphi}(\tau) = e^{-\lambda \tau} \left( H(u(\tau), y(\tau), \nabla V(y(\tau))) - \lambda \Phi(y(\tau)) \right).
$$

In particular: $\dot{\varphi}(0) = H(u(0), x, \nabla \Phi(x)) - \lambda \Phi(x)$ (if $u$ is continuous at 0).
**Proof.** To simplify, we only consider the case where $u$ is continuous, so that $y$ is $C^1$ and $\varphi$ is $C^1(\mathbb{R}^n)$. We have then:

\[
\dot{\varphi}(\tau) = e^{-\lambda \tau} \ell(u(\tau), y(\tau)) + e^{-\lambda \tau} \langle \nabla \Phi(y(\tau)), \dot{y}(\tau) \rangle \\
- \lambda e^{-\lambda \tau} \Phi(y(\tau))
\]

\[
= e^{-\lambda \tau} \left[ \ell(u(\tau), y(\tau)) + \langle \nabla \Phi(y(\tau)), f(u(\tau), y(\tau)) \rangle \right] \\
- \lambda e^{-\lambda \tau} \Phi(y(\tau))
\]

\[
= e^{-\lambda \tau} \left[ H(u(\tau), y(\tau), \nabla \Phi(y(\tau))) - \lambda \Phi(y(\tau)) \right].
\]
HJB in the classical sense

**Theorem 14**

Let $x \in \mathbb{R}^n$. Assume that

- $V$ is continuously differentiable in a neighborhood of $x$
- $P(x)$ has a solution $\bar{u}$ which is continuous at time 0.

Then,

\[
\lambda V(x) - \mathcal{H}(x, \nabla V(x)) = 0,
\]

\[
\bar{u}(0) \in \arg\min_{u_0 \in U} H(u_0, x, \nabla V(x)).
\]
HJB in the classical sense

Proof. Step 1. Let \( u_0 \in U \), let \( u \) be the constant control equal to \( u_0 \), let \( y = y[u, x] \). By the dynamic programming principle, we have:

\[
0 \leq \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(u(t), y(t)) \, dt + e^{-\lambda \tau} V(y(\tau)) - V(x),
\]

for all \( \tau \). Since \( \varphi(0) = 0 \), we deduce from (*) that:

\[
0 \leq \varphi(0) = H(u_0, x, \nabla V(x)) - \lambda V(x).
\]

Therefore,

\[
0 \leq H(u_0, x, \nabla V(x)) - \lambda V(x), \quad \forall u_0 \in U.
\]
Step 2. Let us apply the dynamic programming principle again. Redefining $\varphi$ and setting $\bar{y} = y[\bar{u}, x]$, we obtain:

$$0 = \varphi(\tau) := \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) \, dt + e^{-\lambda \tau} V(\bar{y}(\tau)) - V(x),$$

for all $\tau \geq 0$. It follows that

$$0 = H(\bar{u}(0), x, \nabla V(x)) - \lambda V(x).$$

Step 3. It follows that

$$H(\bar{u}(0), x, \nabla V(x)) = \lambda V(x) \leq H(u_0, x, \nabla V(x)), \quad \forall u_0 \in U.$$

Therefore, $H(\bar{u}(0), x, \nabla V(x)) = \mathcal{H}(x, \nabla V(x))$. 

HJB in the classical sense
Corollary 15

Let $t \geq 0$, assume that $\bar{u}$ is continuous in a neighborhood of $t$ and that $V$ is $C^1$ in a neighborhood of $\bar{y}(t)$, where $\bar{y} := y[\bar{u}, x](t)$. Then,

$$\bar{u}(t) \in \arg\min_{u_0 \in U} H(u_0, \bar{y}, \nabla V(\bar{y})).$$
**Remarks.**
Let us define the Q-function by \( Q(u, y) := H(u, y, \nabla V(y)) \), assuming that \( V \in C^1(\mathbb{R}^n) \).

- If the minimizer is unique in the following relation, we have a **feedback law**:
  \[
  \bar{u}(t) = \arg\min_U Q(\cdot, \bar{y}(t)).
  \]

- In some cases, one can show that \( \nabla V(\bar{y}(t)) = p(t) \), where \( p \) is defined by some adjoint equation → **Pontryagin’s principle**.

- In **Reinforcement Learning**, the approximation of \( Q \) is a central objective.
We will call the equation

$$\lambda v(x) - \mathcal{H}(x, \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n$$

(HJB)

**Hamilton-Jacobi-Bellman** equation, with unknown $v: \mathbb{R}^n \rightarrow \mathbb{R}$.

**Remarks.**

- In general $V$ is not differentiable $\rightarrow$ in **which sense** is the HJB equation to be understood?
- In Theorem 14, we have shown that

$$\bar{u}(t) \in \arg\min H(u_0, y[\bar{u}(t), x], \nabla V(y[\bar{u}(t), x])),$$

(under restrictive assumptions). We will see next that this necessary condition is also **sufficient**.
Theorem 16 (Verification)

Let us assume the assumptions of Theorem 14 hold for all \( x \in \mathbb{R}^n \), so that the HJB equation is satisfied in the classical sense. Let \( x \in \mathbb{R}^n \). Assume that there exists a control \( \bar{u} \) such that

\[
\bar{u}(t) \in \arg\min_{u_0 \in U} H(u_0, \bar{y}(t), \nabla V(\bar{y}(t))),
\]

where \( \bar{y} = y[\bar{u}, x] \). Then \( \bar{u} \) is globally optimal.
Proof. Consider the function:

\[ \varphi(\tau) = \int_0^\tau e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) \, dt + e^{-\lambda \tau} V(\bar{y}(\tau)) - V(x). \]

We have \( \varphi(0) = 0 \). Using (\ast) and Theorem 14, we obtain:

\[ \dot{\varphi}(\tau) = e^{-\lambda \tau} \left[ H(\bar{u}(\tau), \bar{y}(\tau), \nabla V(\bar{y}(\tau)) - V(\bar{y}(\tau))) \right] \]
\[ = e^{-\lambda \tau} \left[ \mathcal{H}(\bar{y}(\tau), \nabla V(\bar{y}(\tau))) - V(\bar{y}(\tau)) \right] \]
\[ = 0. \]

Thus \( \varphi \) is constant, equal to 0. Its limit is given by:

\[ 0 = \int_0^\infty e^{-\lambda t} \ell(\bar{u}(t), \bar{y}(t)) \, dt - V(x) = W(x, \bar{u}) - V(x), \]

proving the optimality of \( \bar{u} \).
1. Introduction

2. Dynamic programming principle

3. A first characterization of the value function

4. HJB equation: the classical sense

5. HJB equation: viscosity solutions
We consider an abstract PDE of the form:

$$F(x, v(x), \nabla v(x)) = 0, \quad \forall x \in \mathbb{R}^n,$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

It contains the HJB equation with

$$F(x, v, p) = \lambda v - \mathcal{H}(x, p).$$

*Goal of the section:* showing that $V$ is a viscosity solution to the HJB equation.
Sub- and super-differentials

Definition 17

Let $v : \mathbb{R}^n \rightarrow \mathbb{R}$. The following sets are called sub- and superdifferential, respectively:

$$D^- v(x) = \left\{ p \in \mathbb{R}^n | \liminf_{y \rightarrow x} \frac{v(y) - v(x) - \langle p, y - x \rangle}{\|y - x\|} \geq 0 \right\}$$

$$D^+ v(x) = \left\{ p \in \mathbb{R}^n | \limsup_{y \rightarrow x} \frac{v(y) - v(x) - \langle p, y - x \rangle}{\|y - x\|} \leq 0 \right\}.$$

Exercise. Let $v(x) = |x|$. Show that $D^- v(0) = [-1, 1]$. 
We have the following characterization.

**Lemma 18**

Let \( v : \mathbb{R}^n \to \mathbb{R} \) be continuous. Let \( p \in \mathbb{R}^n \).

- \( p \in D^- v(x) \iff \) there exists \( \Phi \in C^1(\mathbb{R}^n) \) such that \( \nabla \Phi(x) = p \) and \( v - \Phi \) has a local minimum in \( x \).
- \( p \in D^+ v(x) \iff \) there exists \( \Phi \in C^1(\mathbb{R}^n) \) such that \( \nabla \Phi(x) = p \) and \( v - \Phi \) has a local maximum in \( x \).

**Proof.** The implication \( \implies \) is admitted. The implication \( \iff \) is left as an exercise.
Remark. In the above lemma, one can choose $\Phi(x) = v(x)$ without loss of generality. Thus, we have:

- $(v - \Phi)$ has a local minimum in $x \iff v - \Phi$ is nonnegative in a neighborhood of $x \iff v$ is locally bounded from below by $\Phi$

- $(v - \Phi)$ has a local maximum in $x \iff v - \Phi$ is nonpositive in a neighborhood of $x \iff v$ is locally bounded from above by $\Phi$

Remark. If $v$ is Fréchet differentiable at $x$, then the sub- and superdifferential are equal to $\{\nabla v(x)\}$. 
Definition 19

Let $v : \mathbb{R}^n \to \mathbb{R}$. We call $v$ a **viscosity subsolution** if

$$\mathcal{F}(x, v(x), p) \leq 0, \quad \forall x \in \mathbb{R}^n, \forall p \in D^+ v(x)$$

or, equivalently, if for all $\Phi \in C^1(\mathbb{R}^n)$ such that $v - \Phi$ has a local maximum in $x$,

$$\mathcal{F}(x, v(x), \nabla \Phi(x)) \leq 0.$$
Viscosity solutions

Definition 20

Let \( \nu : \mathbb{R}^n \to \mathbb{R} \). We call \( \nu \) a \textbf{viscosity supersolution} if

\[
\mathcal{F}(x, \nu(x), p) \geq 0, \quad \forall p \in D^- \nu(x)
\]

or, equivalently, if for all \( \Phi \in C^1(\mathbb{R}^n) \) such that \( \nu - \Phi \) has a local minimum in \( x \),

\[
\mathcal{F}(x, \nu(x), \nabla \Phi(x)) \geq 0.
\]

We call \( \nu \) a \textbf{viscosity solution} if it is a sub- and a supersolution.
Viscosity solutions

Theorem 21

The value function $V$ is a viscosity solution of the HJB equation.

Step 1: $V$ is a subsolution. Let $x \in \mathbb{R}^n$, let $\Phi \in C^1(\mathbb{R}^n)$ be such that $V - \Phi$ has a local maximizer in $x$ and $V(x) = \Phi(x)$. We have to prove that

$$\lambda v(x) - H(x, \nabla \Phi(x)) \leq 0.$$ 

Let $u_0 \in U$, let $u$ be the constant control equal to $u_0$ and let $y = y[u, x]$. By the DPP, we have:

$$V(x) \leq \int_0^\tau e^{-\lambda t} \ell(u_0, y(t)) \, dt + e^{-\lambda \tau} V(y(\tau)).$$ 

If $\tau$ is sufficiently small, we have $V(y(\tau)) \leq \Phi(y(\tau))$. 
Viscosity solutions

This implies that for $\tau$ sufficiently small,

$$0 \leq \int_{0}^{\tau} e^{-\lambda t} \ell(u_0, y(t)) \, dt + e^{-\lambda \tau} \Phi(y(\tau)) - \Phi(x) =: \varphi(\tau).$$

Since $\varphi(0) = 0$, we deduce with ($\ast$) that

$$0 \leq \varphi(0) = H(u_0, x, \nabla \Phi(x)) - \lambda V(x).$$

Minimizing with respect to $u_0 \in U$, we obtain:

$$0 \leq \mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x),$$

as was to be proved.
Step 2: $V$ is supersolution. Let $x \in \mathbb{R}^n$, let $\Phi \in C^1(\mathbb{R}^n)$ be such that $V - \Phi$ has a local minimizer in $x$ and such that $V(x) = \Phi(x)$. We have to prove that

$$\lambda v(x) - \mathcal{H}(x, \nabla \Phi(x)) \leq 0.$$  

It follows from the dynamic programming principle that for $\tau > 0$ small enough

$$\Phi(x) \geq \inf_{u \in U_\tau} \int_0^\tau e^{-\lambda t} \ell(u(t), y[x, u](t)) \, dt + e^{-\lambda \tau} \Phi(y[x, u](\tau)).$$

$$=: \phi[u](\tau)$$
Thus by Lemma 13,

\[ 0 \geq \inf_{u \in \mathcal{U}_\infty} \int_0^T \dot{\phi}[u](t) \, dt \]

\[ = \inf_{u \in \mathcal{U}_\infty} \int_0^T e^{-\lambda t} (H(u(t), y[u](t), \nabla \Phi(y[u](t))) - \lambda \Phi(y[u](t)) \, dt \]

\[ \geq \inf_{u \in \mathcal{U}_\infty} \int_0^T e^{-\lambda t} (\mathcal{H}(y[u](t), \nabla \Phi(y[u](t))) - \lambda \Phi(y[u](t)) \, dt. \]

Dividing by \( \tau \) and passing to the limit as \( \tau \downarrow 0 \), we obtain (exercise!) that

\[ 0 \geq \mathcal{H}(x, \nabla \Phi(x)) - \lambda V(x). \]
Theorem 22 (Comparison principle)

Let $v_1$ be a subsolution to the HJB equation. Let $v_2$ be a supersolution to the HJB equation. Then

$$v_1(x) \leq v_2(x), \quad \forall x \in \mathbb{R}^n.$$  

Proof: admitted.

Corollary 23

The value function $V$ is the unique viscosity solution.

Proof. By the comparison principle, any viscosity solution $v$ is such that $v \leq V$ and $v \geq V.$