Optimal control of ODEs Exercise sheet

Laurent Pfeiffer, Inria and CentraleSupélec

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Exercise 1 We consider the following problem:

$$\inf_{\substack{T \ge 0 \\ y \in W^{1,\infty}(0,T;\mathbb{R}^2) \\ u \in L^{\infty}(0,T;\mathbb{R})}} T, \quad \text{subject to} : \begin{cases} \dot{y}_1(t) = u(t) \\ \dot{y}_2(t) = -y_2(t) + u(t) \\ (y_1(0), y_2(0)) = (x_1, x_2) \\ (y_1(T), y_2(T)) = (0, 0) \\ u(t) \in [-1, 1], \end{cases}$$

for a given initial condition $(x_1, x_2) \in \mathbb{R}^2$. We will denote by y[x, u] the solution to the costate equation for a given initial condition x and a given control u.

- 1. Let us assume the existence of a feasible triplet (T, y, u). Justify that the problem possesses a solution.
- 2. Let us fix a solution (T, \bar{y}, \bar{u}) to the problem. Write Pontryagin's principle (we denote by (\bar{p}_1, \bar{p}_2) the associated costate).
- 3. Find an explicit expression of (\bar{p}_1, \bar{p}_2) in function of $(\bar{p}_1(T), \bar{p}_2(T))$.
- 4. Prove that \bar{u} is piecewise constant, with at most two pieces, and that $\bar{u}(t) \in \{-1, 1\}$ for a.e. $t \in [0, T]$.
- 5. Find an explicit expression of y[x, u] for u constant equal to 1 and for u constant equal to -1.
- 6. Compute the following sets:

$$\Gamma_1 = \left\{ x \in \mathbb{R}^2 \mid \exists T \ge 0, \ y[x, u = 1](T) = (0, 0) \right\},\$$

$$\Gamma_{-1} = \left\{ x \in \mathbb{R}^2 \mid \exists T \ge 0, \ y[x, u = 1](T) = (0, 0) \right\}.$$

7. On a graph, draw (approximatively) Γ_1 and Γ_{-1} . Draw the optimal trajectories of the problem for a set of different initial conditions.

Exercise 2

- Consider the lunar landing problem introduced in the lecture, with target set $C = \{(0, y_f)\}$, where $y_f > 0$. Find graphically the set of Pontryagin trajectories (for arbitrary initial conditions.
- Consider the problem introduced in Exercise 1, with target set $C = \{(0, y_f)\}$, where $y_f \in (0, 1)$. Find graphically the set of Pontryagin trajectories (for arbitrary initial conditions.
- Consider the problem introduced in Exercise 1, with target set $C = \{(0, y_f)\}$, where $y_f \in [1, +\infty)$. Find graphically the set of Pontryagin trajectories (for arbitrary initial conditions.

Exercise 3 Consider the following optimal control problem:

$$\inf_{\substack{u \in L^2(0,1)\\ y \in H^1(0,1)}} \int_0^1 \frac{1}{2} y(t)^2 + \frac{1}{6} u(t)^2 \, \mathrm{d}t \quad \text{subject to:} \quad \begin{cases} \dot{y}(t) = y(t) + u(t)\\ y(0) = 1. \end{cases}$$

- 1. Calculate the pre-Hamiltonian and its derivatives.
- 2. Justify the existence of a unique solution to the problem.

- 3. Write the optimality conditions.
- 4. Let (y, u) denote the solution, with associated costate p. Let us set

$$z_1(t) = y(t) - p(t)$$
 and $z_2(t) = y(t) + 3p(t)$.

Show that z_1 and z_2 are solutions to independent linear differential equations. Compute $z_1(t)$ and $z_2(t)$ in function of $z_1(0)$ and $z_2(0)$.

- 5. Compute y and p in function of p(0).
- 6. Formulate the shooting equation and solve it.

Exercise 4 We consider the following problem:

$$\inf_{\substack{u \in L^2(0,T;\mathbb{R}^m)\\y \in H^1(0,T;\mathbb{R}^n)}} \int_0^T \frac{1}{2} \langle y, Wy \rangle + \langle w, y \rangle + \frac{1}{2} \|u(t)\|^2 \mathrm{d}t \quad \text{subject to:} \begin{cases} \dot{y}(t) = Ay(t) + Bu(t)\\ y(0) = y_0. \end{cases}$$

Data: $T > 0, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, W \in \mathbb{R}^{n \times n}, w \in \mathbb{R}^{n}, y_{0}$. The matrix W is assumed to be symmetric positive semi-definite. Given $u \in L^{2}(0,T;\mathbb{R}^{m})$, denote y[u] the solution to the state equation. Let us denote by J the cost function of the optimal control problem, obtained after elimination of the state variable y.

We recall two inequalities, useful for this exercise:

$$ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$$
 and $(a+b)^2 \le 2a^2 + 2b^2$, $\forall a \in \mathbb{R}, \ \forall b \in \mathbb{R}.$

We recall that there exists a constant $M_1 > 0$ such that for all $u \in L^2(0,T;\mathbb{R}^m)$,

$$\|y[u]\|_{L^{\infty}(0,T;\mathbb{R}^n)}^2 \le M_1(\|y_0\|^2 + \|u\|_{L^2(0,T;\mathbb{R}^m)}^2).$$

1. Prove the existence of a constant $M_2 > 0$, independent of y_0 and w such that

$$J(0) \le M_2 (\|y_0\|^2 + \|w\|^2).$$

2. Let $u \in L^2(0,T;\mathbb{R}^m)$ and let y = y[u]. Prove the existence of a constant $M_3 > 0$, depending on T only, such that

$$J(u) \ge \frac{1}{2} \|u\|_{L^{2}(0,T;\mathbb{R}^{m})}^{2} - \frac{M_{3}}{\varepsilon} \|w\|^{2} - \varepsilon M_{3} \|y\|_{L^{\infty}(0,T;\mathbb{R}^{n})}^{2}, \quad \forall \varepsilon > 0.$$

3. Let $u \in L^2(0,T;\mathbb{R}^m)$ be such that $J(u) \leq J(0)$. Prove the existence of $M_4 > 0$, independent of u, y_0 , and w, such that

$$||u||_{L^{2}(0,T;\mathbb{R}^{m})}^{2} \leq M_{4}(||y_{0}||^{2} + ||w||^{2})$$

4. Prove the existence and uniqueness of a solution to the problem.

Exercise 5 This exercise is the continuation of the previous one. We denote by \bar{u} the unique solution to the problem and by \bar{y} the associated trajectory.

- 1. Calculate the Fréchet derivative of J. Find the Riesz representative of $DJ(\bar{u})$.
- 2. Write the optimality system associated with the problem. Denote by p the adjoint variable. The decoupling technique seen in the lecture can be extended to the new setting. To this purpose, we introduce a function $E: [0,T] \to \mathbb{R}^{n \times n}$ and we set:

$$r = p + Ey$$

Find an expression of \dot{r} which does not depend on p. Propose an appropriate Riccati equation for the variable E, so that the system can be decoupled.