Optimal control of ODEs

Exercise sheet

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Exercise 1 We consider the following problem:

$$\inf_{\substack{T \geq 0 \\ y \in W^{1,\infty}(0,T;\mathbb{R}^2) \\ u \in L^\infty(0,T;\mathbb{R})}} T, \quad \text{subject to : } \begin{cases} \dot{y}_1(t) = u(t) \\ \dot{y}_2(t) = -y_2(t) + u(t) \\ (y_1(0), y_2(0)) = (x_1, x_2) \\ (y_1(T), y_2(T)) = (0,0) \\ u(t) \in [-1, 1], \end{cases}$$

for a given initial condition $(x_1, x_2) \in \mathbb{R}^2$. We will denote by $y[x,u]$ the solution to the costate equation for a given initial condition x and a given control u.

1. Let us assume the existence of a feasible triplet (T, y, u). Justify that the problem possesses a solution.
2. Let us fix a solution $(T, \bar{y}, \bar{u})$ to the problem. Write Pontryagin's principle (we denote by $(\bar{p}_1, \bar{p}_2)$ the associated costate).
3. Find an explicit expression of $(\bar{p}_1, \bar{p}_2)$ in function of $(\bar{p}_1(T), \bar{p}_2(T))$.
4. Prove that $\bar{u}$ is piecewise constant, with at most two pieces, and that $\bar{u}(t) \in \{-1, 1\}$ for a.e. $t \in [0, T]$.
5. Find an explicit expression of $y[x,u]$ for u constant equal to 1 and for u constant equal to -1.
6. Compute the following sets:
$$\Gamma_1 = \big\{ x \in \mathbb{R}^2 \,|\, \exists T \geq 0,\, y[x, u=1](T) = (0,0) \big\},$$
$$\Gamma_{-1} = \big\{ x \in \mathbb{R}^2 \,|\, \exists T \geq 0,\, y[x, u=1](T) = (0,0) \big\}.$$
7. On a graph, draw (approximatively) Γ_1 and Γ_{-1}. Draw the optimal trajectories of the problem for a set of different initial conditions.

Exercise 2 Consider the following optimal control problem:

$$\inf_{\substack{u \in L^2(0,1) \\ y \in H^1(0,1)}} \int_0^1 \frac{1}{2} y(t)^2 + \frac{1}{6} u(t)^2 \, \mathrm{d}t \quad \text{subject to: } \begin{cases} \dot{y}(t) = & y(t) + u(t) \\ y(0) = & 1. \end{cases}$$

1. Calculate the pre-Hamiltonian and its derivatives.
2. Justify the existence of a unique solution to the problem.
3. Write the optimality conditions.
4. Let (y, u) denote the solution, with associated costate p. Let us set
$$z_1(t) = y(t) - p(t) \quad \text{and} \quad z_2(t) = y(t) + 3p(t).$$
Show that z_1 and z_2 are solutions to independent linear differential equations. Compute $z_1(t)$ and $z_2(t)$ in function of $z_1(0)$ and $z_2(0)$.
5. Compute y and p in function of $p(0)$.
6. Formulate the shooting equation and solve it.