Optimal control of ODEs

Exercise sheet

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September 28, 2023

Exercise 3 We consider the following problem:

$$\inf_{\substack{u \in L^2(0,T;\mathbb{R}^m) \\ y \in H^1(0,T;\mathbb{R}^n)}} \int_0^T \frac{1}{2}\langle y(t), Wy(t)\rangle + \langle w, y(t)\rangle + \frac{1}{2}\|u(t)\|^2 \mathrm{d}t \quad \text{subject to:} \quad \begin{cases} \dot{y}(t) = & Ay(t) + Bu(t) \\ y(0) = & y_0. \end{cases}$$

Data: $T > 0$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{n\times n}$, $w \in \mathbb{R}^n$, y_0. The matrix W is assumed to be symmetric positive semi-definite. Given $u \in L^2(0,T;\mathbb{R}^m)$, denote $y[u]$ the solution to the state equation. Let us denote by J the cost function of the optimal control problem, obtained after elimination of the state variable y.

We recall two inequalities, useful for this exercise:

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \quad \text{and} \quad (a+b)^2 \leq 2a^2 + 2b^2, \quad \forall a \in \mathbb{R},\ \forall b \in \mathbb{R}.$$

We recall that there exists a constant $M_1 > 0$ such that for all $u \in L^2(0,T;\mathbb{R}^m)$,

$$\|y[u]\|^2_{L^\infty(0,T;\mathbb{R}^n)} \leq M_1\big(\|y_0\|^2 + \|u\|^2_{L^2(0,T;\mathbb{R}^m)}\big).$$

1. Prove the existence of a constant $M_2 > 0$, independent of y_0 and w such that
$$J(0) \leq M_2\big(\|y_0\|^2 + \|w\|^2\big).$$
2. Let $u \in L^2(0,T;\mathbb{R}^m)$ and let $y = y[u]$. Prove the existence of a constant $M_3 > 0$, depending on T only, such that
$$J(u) \geq \frac{1}{2}\|u\|^2_{L^2(0,T;\mathbb{R}^m)} - \frac{M_3}{\varepsilon}\|w\|^2 - \varepsilon M_3 \|y\|^2_{L^\infty(0,T;\mathbb{R}^n)}, \quad \forall \varepsilon > 0.$$
3. Let $u \in L^2(0,T;\mathbb{R}^m)$ be such that $J(u) \leq J(0)$. Prove the existence of $M_4 > 0$, independent of u, y_0, and w, such that
$$\|u\|^2_{L^2(0,T;\mathbb{R}^m)} \leq M_4\big(\|y_0\|^2 + \|w\|^2\big).$$
4. Prove the existence and uniqueness of a solution to the problem.

Exercise 4 Consider the optimal control problem under investigation in the lecture on linear quadratic problems. Let $(y,p) \in H^1(0,T;\mathbb{R}^n)^2$. Then y is the optimal trajectory and p is the associated costate if and only if (y,p) solves

$$\begin{aligned} \dot{y}(t) &= Ay(t) - BB^\top p(t) \qquad & \dot{p}(t) &= -A^\top p(t) - Wy(t) \\ y(0) &= y_0 & p(T) &= Ky(T). \end{aligned}$$

We denote by $\bar{y}$ the unique optimal trajectory and by $\bar{p}$ the associated costate.

1. Let $E\colon [0,T] \to \mathbb{R}^{n\times n}$ be a continuously differentiable matrix-valued function. Let $r\colon [0,T] \to \mathbb{R}^n$ be defined by $r(t) = \bar{p}(t) + E(t)\bar{y}(t)$, for any $t \in [0,T]$. Find an expression of $\dot{r}$ involving the problem data, r, $\bar{y}$, E, and $\dot{E}$ (in other words, an expression of $\dot{r}$ that does not involve p).
2. Now we define E as the solution to the following nonlinear ODE, called Riccati equation:
$$\begin{aligned} -\dot{E}(t) &= E(t)A + A^\top E(t) - W + E(t)BB^\top E(t) \\ E(T) &= -K. \end{aligned}$$
The existence and uniqueness of the solution to the Riccati equation is assumed here (it is not trivial since the expression $E(t)BB^\top E(t)$ is not Lipschitz with respect to $E(t)$). Show that r, as defined in the first item, is null.
3. Propose a method for solving the optimal control problem, relying on the Riccati equation.